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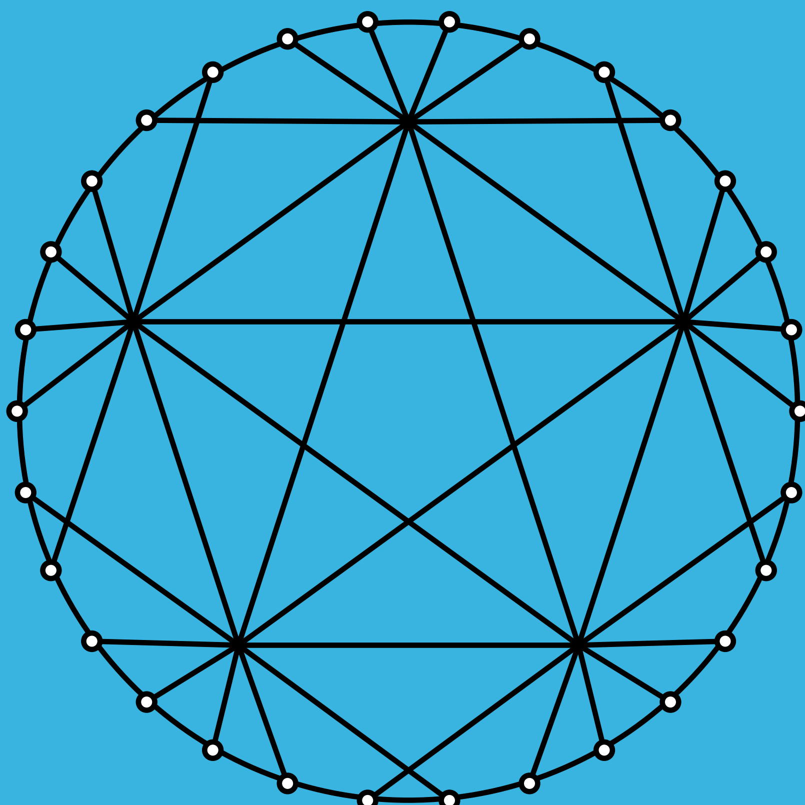

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Ascending Ramsey sequences in graphs

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Abstract. For positive integers m and t with $m \geq t$, let G be a graph with m edges each of which is assigned one of t colors, where each color is assigned to at least one edge. For such a t-edge coloring c of G, an ascending Ramsey sequence in G with respect to c is a sequence G_1, G_2, …, G_k of pairwise edge-disjoint subgraphs of G such that each subgraph G_i $(1 \leq i \leq k)$ is monochromatic and G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq k-1$. The ascending Ramsey index $AR_c(G)$ of G with respect to c is the maximum length of an ascending Ramsey sequence in G with respect to c. The ascending t-Ramsey index $AR_t(G)$ of G is the minimum value of $AR_c(G)$ among all t-edge colorings c of G. With the aid of results on partitions of sets, the exact value of $AR_t(G)$ is determined when G is a star or a matching for every integer $t \geq 2$. Additional results on this topic are presented and some problems in this area are stated.

1 Introduction

More than 35 years ago, a concept and a conjecture were introduced in [1] that has drawn the attention of researchers in graph theory. Let G be a nonempty graph of size m. Thus, $\binom{k+1}{2} \leq m < \binom{k+2}{2}$ for some positive integer k. The graph G is said to have an *ascending subgraph decomposition* $\{G_1, G_2, \ldots, G_k\}$ of k pairwise edge-disjoint subgraphs of G if G_i is isomorphic to a proper subgraph of G_{i+1} for $i = 1, 2, \ldots, k-1$. The following conjecture was stated.

The Ascending Subgraph Decomposition Conjecture.
Every graph has an ascending subgraph decomposition.

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When the famous mathematician Paul Erdős became aware of this conjecture, he immediately doubted its truth and offered $5 for a counterexample or a proof if it turned out to be true. No proof or disproof of this conjecture has ever been given. If the conjecture was shown to be false, then the question occurred of determining the maximum length ℓ of a sequence G_1, G_2, …, G_ℓ of ℓ pairwise edge-disjoint subgraphs of G such that G_i has size i for $1 \leq i \leq \ell$ and G_i is isomorphic to a proper subgraph of G_{i+1} for $i = 1, 2, \ldots, \ell - 1$. This conjecture has been verified for several classes of graphs (see [4, 5, 6, 7, 8, 9], for example). In particular, it is easy to see that a star or matching of size m, where $\binom{k+1}{2} \leq m < \binom{k+2}{2}$, has an ascending subgraph decomposition (into k subgraphs).

While a typical graph theory problem in Ramsey theory deals with determining for given graphs $G_1, G_2, \ldots, G_k$ (often $k = 2$) whether an arbitrary edge coloring of some graph H (often a complete graph) with a fixed number (often two) of colors always results in a monochromatic subgraph isomorphic to some G_i ($1 \leq i \leq k$), we investigate a Ramsey-type concept here inspired by ascending subgraph decompositions.

Let G be a graph (without isolated vertices) of size m with a red-blue edge coloring c. An *ascending Ramsey sequence* of G *with respect to* c is a sequence G_1, G_2, …, G_k of pairwise edge-disjoint subgraphs of G such that each subgraph G_i ($1 \leq i \leq k$) is monochromatic and G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq k-1$. The maximum length of an ascending Ramsey sequence of G with respect to c is the *ascending Ramsey index* $AR_c(G)$ of G. The *ascending Ramsey index* $AR(G)$ of G is

$$AR(G) = \min\{AR_c(G) : \ c \text{ is a red-blue edge coloring of } G\}.$$

These concepts were introduced in [2, 3]. Among the results presented in [2, 3] is a characterization of all stars and matchings G for which $AR(G) = k$ for every integer $k \geq 2$. In particular, the following result was obtained.

Theorem 1.1. [3] *Let $k \geq 2$ be an integer and let G be a star $K_{1,m}$ or a matching mK_2. Then $AR(G) = k$ if and only if*

$$\binom{k+1}{2} \leq m < \binom{k+2}{2}.$$

This result shows that not only does a star or matching of size m have an ascending subgraph decomposition but for every 2-edge coloring of these graphs, there is an ascending subgraph decomposition in which each subgraph is monochromatic.

A goal here is to generalize ascending Ramsey index from an edge coloring employing two colors to an edge coloring with at least two colors resulting in a multicolor ascending Ramsey index and determine its value when the graph is a star or a matching. In Section 2, we introduce the concepts of ascending subset sequence and index of a set and present results on partitioning a given set with certain prescribed ascending properties. In Section 3, we apply the results from Section 2 on set partitions to present a characterization of those stars and matchings G for which $AR_t(G) = k$ for all integers $t \geq 2$ and $k \geq 2$. A more general problem dealing with this topic is also discussed in Section 3.

2 Ascending subset sequences

Prior to defining the multicolor ascending Ramsey index of a graph G and then determining its value when G is a star or a matching, it is useful to consider and solve a problem involving sets, which is perhaps an interesting set theory problem in its own right.

> For a finite set S with $|S| = m \geq 2$ and a partition $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ of S into t subsets (a t-partition of S) where $m \geq t \geq 2$, what is the maximum number k of pairwise disjoint subsets S_i, $1 \leq i \leq k$, of S such that (1) $|S_i| = i$ for $1 \leq i \leq k$ and (2) for every integer i with $1 \leq i \leq k$, there exists an integer j with $1 \leq j \leq t$ for which $S_i \subseteq T_j$?

For a set S with $|S| = m \geq 2$ and a partition $\mathcal{P}$ of S, a sequence S_1, S_2, …, S_k of k pairwise disjoint subsets of S that satisfies (1) and (2) is called an *ascending subset sequence* of S with respect to $\mathcal{P}$ and the maximum length of an ascending subset sequence of S with respect to $\mathcal{P}$ is the *ascending subset index* $AR_{\mathcal{P}}(S)$ of S with respect to $\mathcal{P}$. For an integer $t \geq 2$, the *ascending subset index* $AR_t(S)$ of S is defined by

$$AR_t(S) = \min\{AR_{\mathcal{P}}(S) : \ \mathcal{P} \text{ is a } t\text{-partition of } S\}.$$

Thus, if S is a set with $AR_t(S) = k \geq 2$, then

$$|S| \geq \max\left\{t,\ \binom{k+1}{2}\right\}. \tag{1}$$

The following lemma will be particularly useful in what follows.

Lemma 2.1. *Let t and k be integers with $t \geq 2$ and $k \geq 2$ and let S and S^* be sets such that $t \leq |S| < |S^*|$. If there exists an ascending subset sequence of length k for every t-partition of S, then there is an ascending subset sequence of length k for every t-partition of S^*. Consequently, $AR_t(S) \leq AR_t(S^*)$.*

Proof. Using a one-to-one mapping from the set S to S^* if necessary, we may assume that $S \subseteq S^*$. Let $\mathcal{P}^* = \{T_1^*, T_2^*, \ldots, T_t^*\}$ be a t-partition of S^* where $t \leq |S| < |S^*|$. Thus, $|S^*| = \sum_{i=1}^{t} |T_i^*|$. Since $t \leq |S| < |S^*|$, there is a t-partition $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ of S such that $T_i \subseteq T_i^*$ for $i = 1, 2, \ldots, t$. By assumption, there is an ascending subset sequence S_1, S_2, …, S_k of S such that $|S_i| = i$ for $1 \leq i \leq k$ and for every integer i with $1 \leq i \leq k$, $S_i \subseteq T_j$ for some integer j with $1 \leq j \leq t$. Since $T_j \subseteq T_j^*$ for $j = 1, 2, \ldots, t$, it follows that S_1, S_2, …, S_k is also an ascending subset sequence of S^* with respect to $\mathcal{P}^*$. □

If S is a set such that $AR_t(S) = k$ for integers $t \geq 2$ and $k \geq 2$, then a question of interest here concerns what can be said about $|S|$. A related question deals with determining whether it is possible for $AR_t(S) = k$ when given information on the value of $|S|$ in terms of t and k. In order to provide answers to these questions, we consider the two possibilities where either $t > k \geq 2$ or $k \geq t \geq 2$. We begin with the situation where $t > k \geq 2$.

Theorem 2.2. *Let S be a set with $AR_t(S) \geq k \geq 2$ where $t \geq 2$.*

$$\text{If } t > k, \text{ then } |S| \geq (k-1)(t-1) + k.$$

Proof. Suppose that $t > k$. We show that if S is a set with $|S| < (k-1)(t-1) + k$, then $AR_t(S) < k$, namely, there is a t-partition $\mathcal{P}$ of S such that there exists no ascending subset sequence of length k in S with respect to $\mathcal{P}$. By Lemma 2.1, we may assume that $|S| = [(k-1)(t-1)+k]-1 = t(k-1)$. Let $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ be a t-partition of S such that $|T_i| = k-1$ for $1 \leq i \leq t$. Then there is no ascending subset sequence $S_1, S_2, \ldots, S_k$ of length k in S with respect to $\mathcal{P}$ since S_k is not a subset of T_i for any integer i with $1 \leq i \leq t$. Consequently, there exists no ascending subset sequence of length k in S with respect to $\mathcal{P}$ and so $AR_t(S) < k$. □

In order to establish the next result, we first present the following lemma.

Lemma 2.3. *For integers $t, k,$ and j with $t > k \geq 2$ and $0 \leq j \leq k$,*

$$\frac{1}{t}\left[kt+1-\binom{k+2}{2}+\binom{k+2-j}{2}\right] > k-j. \tag{2}$$

Proof. Since $2j(t-k)+(j-1)(j-2) > 0$ when $j \geq 0$ and $t > k$, it follows that

$$\begin{aligned} kt+1-&\binom{k+2}{2}+\binom{k+2-j}{2} \\ &= kt+1-\frac{(k+2)(k+1)}{2}+\frac{(k+2-j)(k+1-j)}{2} > kt-jk \end{aligned}$$

and so (2) holds. □

Theorem 2.4. *Let t and k be integers with $t > k \geq 2$. If S is a set with $AR_t(S) = k$, then*

$$(k-1)(t-1)+k \leq |S| \leq kt.$$

Proof. Since $AR_t(S) = k < t$, it follows that $|S| \geq (k-1)(t-1)+k$ by Theorem 2.2. It remains therefore to show that $|S| \leq kt$. Assume, to the contrary, that there is a set S with $AR_t(S) = k < t$ such that $|S| \geq kt+1$. We show that for every t-partition $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ of S, there exists an ascending subset sequence of length $k+1$ with respect to $\mathcal{P}$, that is, a sequence $S_1, S_2, \ldots, S_{k+1}$ of $k+1$ pairwise disjoint subsets of S such that (1) $|S_i| = i$ for $1 \leq i \leq k+1$ and (2) for every integer i with $1 \leq i \leq k+1$, there exists an integer j with $1 \leq j \leq t$ for which $S_i \subseteq T_j$. This, however, implies that $AR_t(S) \geq k+1$, which contradicts the assumption that $AR_t(S) = k$.

By Lemma 2.1, we may assume that $|S| = kt+1$. Let $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ be a t-partition of S. Since $\frac{1}{t}(kt+1) > k$, it follows that at least one of the subsets $T_1, T_2, \ldots, T_t$ contains a $(k+1)$-element subset S_{k+1}, say $S_{k+1} \subseteq T_{i_1}$ where $1 \leq i_1 \leq t$. For $i = 1, 2, \ldots, t$, where $i \neq i_1$, let $T_i^{(1)} = T_i$ and let $T_{i_1}^{(1)} = T_{i_1} - S_{k+1}$ (where $T_{i_1}^{(1)}$ may be empty). Then $\mathcal{P}^{(1)} = \left\{T_1^{(1)}, T_2^{(1)}, \ldots, T_t^{(1)}\right\}$ is a collection of t pairwise disjoint subsets of the set $S^{(1)} = S - S_{k+1}$, whose union is $S^{(1)}$. Observe that

$$|S^{(1)}| = |S| - |S_{k+1}| = (kt+1)-(k+1) = (kt+1)-\binom{k+2}{2}+\binom{k+1}{2}.$$

Since

$$\frac{1}{t}\left[kt+1-\binom{k+2}{2}+\binom{k+1}{2}\right]>k-1,$$

by Lemma 2.3, at least one of the subsets $T_1^{(1)},T_2^{(1)},\ldots,T_t^{(1)}$ contains a k-element subset S_k, say $S_k\subseteq T_{i_2}$ where $1\le i_2\le t$. For $i=1,2,\ldots,t$ where $i\ne i_2$, let $T_i^{(2)}=T_i^{(1)}$ and let $T_{i_2}^{(2)}=T_{i_2}^{(1)}-S_k$ (where $T_{i_2}^{(2)}$ may be empty). Then $\mathcal{P}^{(2)}=\left\{T_1^{(2)},T_2^{(2)},\ldots,T_t^{(2)}\right\}$ is a collection of t pairwise disjoint subsets of the set $S^{(2)}=S^{(1)}-S_k$, whose union is $S^{(2)}$. Observe that

$$\begin{aligned}|S^{(2)}| &= |S^{(1)}|-|S_k|=\left[(kt+1)-\binom{k+2}{2}+\binom{k+1}{2}\right]-k\\ &= (kt+1)-\binom{k+2}{2}+\left[\binom{k+1}{2}-k\right]\\ &= (kt+1)-\binom{k+2}{2}+\binom{k}{2}.\end{aligned}$$

Repeating this procedure recursively, we obtain a sequence $\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$, …, $\mathcal{P}^{(k)}$ where $\mathcal{P}^{(j)}=\left\{T_1^{(j)},T_2^{(j)},\ldots,T_t^{(j)}\right\}$ is a collection of t pairwise disjoint subsets of the set $S^{(j)}=S^{(j-1)}-S_{k+2-j}$, whose union is $S^{(j)}$ for $1\le j\le k$ and $S^{(0)}=S$. For each integer j with $1\le j\le k$, observe that

$$\begin{aligned}|S^{(j)}| &= |S^{(j-1)}|-|S_{k+2-j}|\\ &= \left[(kt+1)-\binom{k+2}{2}+\binom{k+2-(j-1)}{2}\right]-(k+2-j)\\ &= \left[(kt+1)-\binom{k+2}{2}+\binom{k+1-j}{2}\right]-(k+2-j)\\ &= (kt+1)-\binom{k+2}{2}+\binom{k+2-j}{2}.\end{aligned}$$

By Lemma 2.3, for $1\le j\le k$,

$$\frac{1}{t}\left[(kt+1)+\binom{k+2}{2}+\binom{k+2-j}{2}\right]>k-j$$

and so at least one of the subsets $T_1^{(j)},T_2^{(j)},\ldots,T_t^{(j)}$ contains a $(k-j+1)$-element subset S_{k-j+1}. By the construction of the sequence $\mathcal{P},\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$, …, $\mathcal{P}^{(k)}$, it follows that $S_1,S_2,\ldots,S_{k+1}$ is an ascending subset sequence of length $k+1$ in S with respect to $\mathcal{P}$, which implies that $AR_t(S)\ge k+1$, a contradiction. □

Notice that the proof of Theorem 2.4 shows the following.

If S is a set with $|S| \geq kt+1$, where $t > k \geq 2$, then $AR_t(S) \geq k+1$. (3)

We now show that for integers t and k when $t > k \geq 2$ that if S is a set with $|S| = (k-1)(t-1)+k$, then $AR_t(S) \geq k$.

Theorem 2.5. *Let t and k be integers with $t > k \geq 2$.*

$$\textit{If } S \textit{ is a set with } |S| = (k-1)(t-1)+k, \textit{ then } AR_t(S) \geq k.$$

Proof. Let S be a set with $|S| = (k-1)(t-1)+k$ and let $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ be a t-partition of S. We show that there exists an ascending subset sequence of length k in S with respect to $\mathcal{P}$, that is, a sequence S_1, S_2, ..., S_k of k pairwise disjoint subsets of S such that (1) $|S_i| = i$ for $1 \leq i \leq k$ and (2) for every integer i with $1 \leq i \leq k$, there exists an integer j with $1 \leq j \leq t$ for which $S_i \subseteq T_j$. Since

$$\frac{1}{t}\left[(k-1)(t-1)+k\right] > k-1,$$

at least one of the subsets $T_1, T_2, \ldots, T_t$ contains a k-element subset S_k. Suppose that $S_k \subseteq T_{i_1}$ where $1 \leq i_1 \leq t$. For $i = 1, 2, \ldots, t$ where $i \neq i_1$, let $T_i^{(1)} = T_i$ and let $T_{i_1}^{(1)} = T_{i_1} - S_k$ (where $T_{i_1}^{(1)}$ may be empty). Then $\mathcal{P}^{(1)} = \left\{T_1^{(1)}, T_2^{(1)}, \ldots, T_t^{(1)}\right\}$ is a collection of t pairwise disjoint subsets of the set $S^{(1)} = S - S_k$, whose union is $S^{(1)}$. Since $t-1 > k-1$, it follows that

$$\begin{aligned} |S^{(1)}| &= |S| - |S_k| = [(k-1)(t-1)+k] - k \\ &= (k-1)(t-1) = (k-2)(t-1) + (t-1) \\ &> (k-2)(t-1) + (k-1). \end{aligned}$$

Since

$$\frac{1}{t}\left[(k-2)(t-1)+(k-1)\right] > k-2,$$

at least one of the subsets $T_1^{(1)}, T_2^{(1)}, \ldots, T_t^{(1)}$ contains a $(k-1)$-element subset S_{k-1}. Suppose that $S_{k-1} \subseteq T_{i_2}^{(1)}$ where $1 \leq i_2 \leq t$. For $i = 1, 2, \ldots, t$ where $i \neq i_2$, let $T_i^{(2)} = T_i^{(1)} =$ and let $T_{i_2}^{(2)} = T_{i_2}^{(1)} - S_{k-1}$ (where $T_{i_2}^{(2)}$ may be empty). Then $\mathcal{P}^{(2)} = \left\{T_1^{(2)}, T_2^{(2)}, \ldots, T_t^{(2)}\right\}$ is a collection of t pairwise disjoint subsets of the set $S^{(2)} = S^{(1)} - S_{k-1}$, whose union is $S^{(2)}$. Since

$$|S^{(1)}| > (k-2)(t-1) + (k-1)$$

and $t-1 > k-1 > k-2$, it follows that

$$\begin{aligned} |S^{(2)}| &= |S^{(1)}| - |S_{k-1}| > [(k-2)(t-1) + (k-1)] - (k-1) \\ &= (k-2)(t-1) = (k-3)(t-1) + (t-1) \\ &> (k-3)(t-1) + (k-2). \end{aligned}$$

Repeating this procedure recursively, we obtain a sequence $\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$, ..., $\mathcal{P}^{(k-1)}$ where $\mathcal{P}^{(j)} = \left\{T_1^{(j)}, T_2^{(j)}, \ldots, T_t^{(j)}\right\}$ is a collection of t pairwise disjoint subsets of $S^{(j)} = S^{(j-1)} - S_{k-j+1}$, whose union is $S^{(j)}$ for $1 \le j \le k$ and $S^{(0)} = S$. For each integer j with $1 \le j \le k-1$, since

$$|S^{(j-1)}| \ge (k-j)(t-1) + (k-j+1)$$

and $t-1 > k-1 \ge k-j$, it follows that

$$\begin{aligned} |S^{(j)}| &= |S^{(j-1)}| - |S_{k-j+1}| \\ &\ge [(k-j)(t-1) + (k-j+1)] - (k-j+1) \\ &= (k-j)(t-1) = (k-j-1)(t-1) + (t-1) \\ &> (k-j-1)(t-1) + (k-j). \end{aligned}$$

Since

$$\frac{1}{t}\left[(k-j-1)(t-1) + (k-j)\right] > k-j-1$$

for $1 \le j \le k-1$, at least one of the subsets $T_1^{(j)}, T_2^{(j)}, \ldots, T_t^{(j)}$ contains a $(k-j)$-element subset S_{k-j}. By the construction of the sequence $\mathcal{P}, \mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$, ..., $\mathcal{P}^{(k-1)}$, it follows that $S_1, S_2, \ldots, S_k$ is an ascending subset sequence of length k in S with respect to $\mathcal{P}$. Therefore, $AR_t(S) \ge k$. $\square$

We saw in (3) that if t and k are integers with $t > k \ge 2$ and S is a set with $|S| \ge kt + 1$, then $AR_t(S) \ge k + 1$. Hence, the following is a consequence of Lemma 2.1 and Theorem 2.5.

Corollary 2.6. *Let t and k be integers with $t > k \geq 2$. If S is a set with*

$$(k-1)(t-1) + k \leq |S| \leq kt,$$

then $AR_t(S) \geq k$.

We are now able to present a necessary and sufficient condition on the size of a set S for which $AR_t(S) = k$ when $t > k \geq 2$.

Theorem 2.7. *Let t and k be integers with $t > k \geq 2$. Then $AR_t(S) = k$ for a set S if and only if*

$$(k-1)(t-1) + k \leq |S| \leq kt.$$

Proof. By Theorem 2.4, if $AR_t(S) = k$, then $(k-1)(t-1)+k \leq |S| \leq kt$. For the converse, suppose that S is a set such that $(k-1)(t-1)+k \leq |S| \leq kt$. By Corollary 2.6, $AR_t(S) \geq k$. It remains to show that $AR_t(S) \leq k$. By Lemma 2.1, we may assume that $|S| = kt$. Let $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ be a t-partition of S such that $|T_i| = k$ for $1 \leq i \leq t$. Then there exists no ascending subset sequence of length $k+1$ in S with respect to $\mathcal{P}$. Hence, $AR_t(S) \leq k$ and so $AR_t(S) = k$. □

Next, we consider the situation where $k \geq t \geq 2$. First, we present a lower bound for the size of a set S having $AR_t(S) \geq k$ where $k \geq t \geq 2$.

Theorem 2.8. *Let S be a set with $AR_t(S) \geq k \geq 2$ where $t \geq 2$.*

$$\text{If } k \geq t, \text{ then } |S| \geq \tbinom{k+1}{2} + \tbinom{t-1}{2}.$$

Proof. Let $k \geq t$. We show that if S is a set with $|S| < \binom{k+1}{2} + \binom{t-1}{2}$, then there is a t-partition $\mathcal{P}$ of S such that there exists no ascending subset sequence of length k in S with respect to $\mathcal{P}$. By Lemma 2.1, we may assume that $|S| = \binom{k+1}{2} + \binom{t-1}{2} - 1$. Since $(t-1)^2 = \binom{t}{2} + \binom{t-1}{2}$ and $\binom{t}{2} < \binom{k+1}{2} - 1$ when $2 \leq t \leq k$, it follows that

$$(t-1)^2 = \binom{t}{2} + \binom{t-1}{2} < \binom{k+1}{2} + \binom{t-1}{2} - 1 = |S|.$$

Let $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ be a t-partition of S such that $|T_i| = t-1$ for $1 \leq i \leq t-1$. Thus, $|T_t| = |S| - (t-1)^2 > 0$. Assume, to the contrary, that there exists an ascending subset sequence $S_1, S_2, \ldots, S_k$ of length k in S

with respect to $\mathcal{P}$. Since $t-1 \leq k-1$, it follows that $S_i \subseteq T_t$ for $t \leq i \leq k$. Hence,

$$|T_t| \geq \sum_{i=t}^{k} |S_i| = \sum_{i=t}^{k} i = \sum_{i=1}^{k} i - \sum_{i=1}^{t-1} i = \binom{k+1}{2} - \binom{t}{2}.$$

Therefore,

$$|S| = \left(\sum_{i=1}^{t-1} |T_i|\right) + |T_t| \geq (t-1)^2 + \binom{k+1}{2} - \binom{t}{2} = \binom{k+1}{2} + \binom{t-1}{2}.$$

This contradicts the assumption that $|S| = \binom{k+1}{2} + \binom{t-1}{2} - 1$. $\square$

Prior to presenting the next result, we establish the following lemma, which will be useful.

Lemma 2.9. *Let $t \geq 2$ be an integer. For each positive integer j,*

$$\frac{1}{t}\left[\binom{j+1}{2} + \binom{t-1}{2}\right] > j-1. \tag{4}$$

Proof. First, observe that for every two integers $t \geq 2$ and $j \geq 1$, it follows that

$$\left[(j-t) + \frac{1}{2}\right]^2 + \frac{7}{4} > 0.$$

Thus, $(j-t)^2 + (j-t) + 2 > 0$ and so

$$(j^2 + j) + (t^2 - 3t + 2) > 2t(j-1).$$

Therefore, $\binom{j+1}{2} + \binom{t-1}{2} > t(j-1)$, giving the desired inequality. $\square$

Theorem 2.10. *Let t and k be integers with $k \geq t \geq 2$. If S is a set with*

$$|S| = \binom{k+1}{2} + \binom{t-1}{2},$$

then $AR_t(S) \geq k$.

Proof. Let S be a set with $|S| = \binom{k+1}{2} + \binom{t-1}{2}$ and let $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ be a t-partition of S. We show that there exists an ascending subset sequence of length k in S with respect to $\mathcal{P}$, that is, there is a sequence S_1, S_2,

..., S_k of k pairwise disjoint subsets of S such that (1) $|S_i| = i$ for $1 \le i \le k$ and (2) for every integer i with $1 \le i \le k$, there exists an integer j with $1 \le j \le t$ for which $S_i \subseteq T_j$. Since

$$\frac{1}{t}\left[\binom{k+1}{2} + \binom{t-1}{2}\right] > k-1$$

by Lemma 2.9, it follows that at least one of the subsets T_i ($1 \le i \le t$) contains a k-element subset S_k. Suppose that $S_k \subseteq T_{i_1}$, where $1 \le i_1 \le t$. Let $S^{(1)} = S - S_k$. For $i = 1, 2, \ldots, t$ where $i \ne i_1$, let $T_i^{(1)} = T_i$ and let $T_{i_1}^{(1)} = T_{i_1} - S_k$ (where $T_{i_1}^{(1)}$ may be empty). Then $\bigcup_{i=1}^{t} T_i^{(1)} = S^{(1)}$ and $\mathcal{P}^{(1)} = \left\{T_1^{(1)}, T_2^{(1)}, \ldots, T_t^{(1)}\right\}$ is a collection of t pairwise disjoint subsets of $S^{(1)}$. Observe that

$$|S^{(1)}| = |S| - |S_k| = \left[\binom{k+1}{2} + \binom{t-1}{2}\right] - k = \binom{k}{2} + \binom{t-1}{2}.$$

Since

$$\frac{1}{t}\left[\binom{(k-1)+1}{2} + \binom{t-1}{2}\right] > k-2$$

by Lemma 2.9, it follows that at least one of the subsets $\mathcal{P}^{(1)}$ contains a $(k-1)$-element subset S_{k-1}. Suppose that $S_{k-1} \subseteq T_{i_2}$ where $1 \le i_2 \le t$. For $i = 1, 2, \ldots, t$ where $i \ne i_2$, let $T_i^{(2)} = T_i^{(1)}$ and let $T_{i_2}^{(2)} = T_{i_2}^{(1)} - S_{k-1}$ (where $T_{i_2}^{(2)}$ may be empty). Repeating this procedure recursively, we obtain a sequence $\mathcal{P}^{(1)}$, $\mathcal{P}^{(2)}$, ..., $\mathcal{P}^{(k-1)}$ where $\mathcal{P}^{(j)} = \left\{T_1^{(j)}, T_2^{(j)}, \ldots, T_t^{(j)}\right\}$ is a collection of t pairwise disjoint subsets of the set $S^{(j)} = S^{(j-1)} - S_{k-j+1}$, where $\bigcup_{i=1}^{t} T_i^{(j)} = S^j$ for $1 \le j \le k-1$ and $S^{(0)} = S$. For each integer j with $1 \le j \le k-1$, observe that

$$\begin{aligned} |S^{(j)}| &= |S^{(j-1)}| - |S_{k-j+1}| \\ &= \left[\binom{(k-j+1)+1}{2} + \binom{t-1}{2}\right] - (k-j+1) \\ &= \binom{(k-j)+1}{2} + \binom{t-1}{2}. \end{aligned}$$

Since

$$\frac{1}{t}\left[\binom{(k-j)+1}{2} + \binom{t-1}{2}\right] > k-j-1$$

by Lemma 2.9, it follows that at least one of the subsets $T_1^{(j)}, T_2^{(j)}, \ldots, T_t^{(j)}$ contains a $(k-j)$-element subset S_{k-j}. By the construction of the sequence $\mathcal{P}, \mathcal{P}^{(1)}, \mathcal{P}^{(2)}, \ldots, \mathcal{P}^{(k-1)}$, it follows that $S_1, S_2, \ldots, S_k$ is an ascending subset sequence of length k in S with respect to $\mathcal{P}$. Therefore, $AR_t(S) \geq k$. □

The following is a consequence of Lemma 2.1 and Theorem 2.10.

Corollary 2.11. *Let t and k be integers with $k \geq t \geq 2$. If S is a set with*

$$\binom{k+1}{2} + \binom{t-1}{2} \leq |S| \leq \binom{k+2}{2} + \binom{t-1}{2} - 1,$$

then $AR_t(S) \geq k$.

We now have the following result.

Theorem 2.12. *Let t and k be integers with $k \geq t \geq 2$. If $AR_t(S) = k$, then*

$$\binom{k+1}{2} + \binom{t-1}{2} \leq |S| \leq \binom{k+2}{2} + \binom{t-1}{2} - 1.$$

Proof. Let S be a set with $AR_t(S) = k \geq t$. By Theorem 2.8, it follows that $|S| \geq \binom{k+1}{2} + \binom{t-1}{2}$. It remains to establish the upper bound for $|S|$. Since $k+1 > t$, it follows from Corollary 2.11 that if $|S| \geq \binom{k+2}{2} + \binom{t-1}{2}$, then $AR_t(S) \geq k+1$. Hence, if $AR_t(S) \leq k$, then $|S| \leq \binom{k+2}{2} + \binom{t-1}{2} - 1$. In particular, if $AR_t(S) = k$, then $|S| \leq \binom{k+2}{2} + \binom{t-1}{2} - 1$. □

We are now able to present a necessary and sufficient condition on the size of a set S for which $AR_t(S) = k$ when $k \geq t \geq 2$.

Theorem 2.13. *Let t and k be integers with $k \geq t \geq 2$. Then $AR_t(S) = k$ for a set S if and only if*

$$\binom{k+1}{2} + \binom{t-1}{2} \leq |S| \leq \binom{k+2}{2} + \binom{t-1}{2} - 1.$$

Proof. By Theorem 2.12, if $AR_t(S) = k$, then

$$\binom{k+1}{2} + \binom{t-1}{2} \leq |S| \leq \binom{k+2}{2} + \binom{t-1}{2} - 1. \tag{5}$$

For the converse, suppose that S is a set such that (5) holds. By Corollary 2.11, $AR_t(S) \geq k$. It remains to show that $AR_t(S) \leq k$. By Lemma 2.1, we may assume that $|S| = \binom{k+2}{2} + \binom{t-1}{2} - 1$ and show that there is a t-partition $\mathcal{P}$ of S such that there exists no ascending subset sequence of length $k+1$ in S with respect to $\mathcal{P}$. Since $(t-1)^2 = \binom{t}{2} + \binom{t-1}{2}$ and $\binom{t}{2} < \binom{k+1}{2} - 1 < \binom{k+2}{2} - 1$ when $2 \le t \le k$, it follows that

$$\begin{aligned}(t-1)^2 &= \binom{t}{2} + \binom{t-1}{2} \\ &< \binom{k+1}{2} + \binom{t-1}{2} - 1 \\ &< \binom{k+2}{2} + \binom{t-1}{2} - 1 \\ &= |S|.\end{aligned}$$

Let $\mathcal{P} = \{T_1, T_2, \ldots, T_t\}$ be a t-partition of S such that $|T_i| = t-1$ for $1 \le i \le t-1$. Thus, $|T_t| = |S| - (t-1)^2 > 0$. Assume, to the contrary, that there exists an ascending subset sequence $S_1, S_2, \ldots, S_{k+1}$ of length $k+1$ in S with respect to $\mathcal{P}$. Since $t-1 \le k-1$, it follows that $S_i \subseteq T_t$ for $t \le i \le k+1$. Hence,

$$|T_t| \ge \sum_{i=t}^{k+1} |S_i| = \sum_{i=t}^{k+1} i = \sum_{i=1}^{k+1} i - \sum_{i=1}^{t-1} i = \binom{k+2}{2} - \binom{t}{2}.$$

Therefore,

$$|S| = \left(\sum_{i=1}^{t-1} |T_i|\right) + |T_t| \ge (t-1)^2 + \binom{k+2}{2} - \binom{t}{2} = \binom{k+1}{2} + \binom{t-1}{2}.$$

This contradicts the assumption that $|S| = \binom{k+2}{2} + \binom{t-1}{2} - 1$. □

3 The multicolor ascending Ramsey index of a graph

We now return to our primary topic. For positive integers m and t with $m \ge t$, let G be a graph (without isolated vertices) with m edges each of which is assigned one of t colors, where each color is assigned to at least one edge. For such a t-edge coloring c of G, an *ascending Ramsey sequence in G with respect to c* is a sequence G_1, G_2, …, G_k of pairwise edge-disjoint subgraphs (without isolated vertices) of G such that (1) each subgraph G_i ($1 \le i \le k$) is monochromatic and (2) G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \le i \le k-1$. The *ascending Ramsey index* $AR_c(G)$ *of* G *with respect to* c is the maximum length of an ascending Ramsey sequence in G with respect to c. Thus, if $AR_c(G) = k$, then there exists an ascending Ramsey sequence $G_1, G_2, \ldots, G_k$ of k subgraphs of G with respect to c where $|E(G_i)| = i$ for $1 \le i \le k$. The *ascending* t*-Ramsey index* $AR_t(G)$ of G is defined by

$$AR_t(G) = \min\{AR_c(G) : \ c \text{ is a } t\text{-edge coloring of } G\}.$$

If $t = 1$, then $AR_1(G)$ returns us to the Ascending Subgraph Decomposition Conjecture of a graph G. Thus, if G is a graph of size m with $AR_t(G) = k \ge 2$, then

$$m \ge \max\left\{ t\,,\ \binom{k+1}{2} \right\}.$$

The goal here is to determine for each pair k, t of integers with $k \ge 2$ and $t \ge 2$ all those integers $m \ge \max\{t, \binom{k+1}{2}\}$ for which $AR_t(G) = k$ when $G = K_{1,m}$ is a star of size m or when $G = mK_2$ is a matching of size m. We saw that this question was answered in [3] when $t = 2$, where then $AR_2(G) = AR(G)$. If $G = K_{1,m}$ or $G = mK_2$ and $S = E(G)$ and S' is a nonempty subset of S where $1 \le |S'| = i \le m$, then the subgraph $G[S']$ of G induced by the set S' of the edges of G is the star $K_{1,i}$ if $G = K_{1,m}$ or the matching iK_2 if $G = mK_2$. Consequently, the results of Theorems 2.7 and 2.13 obtained in Section 2 give us an immediate answer.

Corollary 3.1. *Let t and k be integers with $2 \le k < t$ and let $G = K_{1,m}$ or $G = mK_2$ where $m \ge \max\{t, \binom{k+1}{2}\}$. Then $AR_t(G) = k$ if and only if*

$$(k-1)(t-1) + k \le m \le kt.$$

Corollary 3.2. *Let t and k be integers with $k \ge t \ge 2$ and let $G = K_{1,m}$ or $G = mK_2$ where $m \ge \max\{t, \binom{k+1}{2}\}$. Then $AR_t(G) = k$ if and only if*

$$\binom{k+1}{2} + \binom{t-1}{2} \le m \le \binom{k+2}{2} + \binom{t-1}{2} - 1.$$

As a consequence of Corollaries 3.1 and 3.2, we have the following. For a real number x, let $\lceil x \rceil$ denote the ceiling of x.

Corollary 3.3. *Let $t \ge 2$ and $m \ge t$ be integers and let $G \in \{K_{1,m}, mK_2\}$.*

(*a*) *If* $m \leq t(t-1)$, *then* $AR_t(G) = k$ *where* k *is the unique positive integer such that* $(k-1)(t-1)+k \leq m \leq kt$. *Thus,*

$$AR_t(G) = \left\lceil \frac{m}{t} \right\rceil.$$

(*b*) *If* $m > t(t-1)$, *then* $AR_t(G) = k$ *where* k *is the unique positive integer such that* $\binom{k+1}{2} + \binom{t-1}{2} \leq m \leq \binom{k+2}{2} + \binom{t-1}{2} - 1$. *Thus,*

$$AR_t(G) = \left\lceil \frac{-3+\sqrt{8m+12t-4t^2+1}}{2} \right\rceil.$$

Proof. Let $AR_t(G) = k$. Then $k < t$ or $k \geq t$. If $k < t$, then $m \leq kt \leq t(t-1)$ by Corollary 3.1; while if $k \geq t$, then $m \geq \binom{k+1}{2} + \binom{t-1}{2} \geq \binom{t+1}{2} + \binom{t-1}{2} > t(t-1)$ by Corollary 3.2. Therefore, if $m \leq t(t-1)$, then $k < t$ and Corollary 3.1 applies. We show $k = \lceil \frac{m}{t} \rceil$. First, since $m \leq kt$, it follows that $k \geq \frac{m}{t}$. Next, since $m \geq (k-1)(t-1)+k$, it follows that

$$\frac{m}{t} \geq \frac{(k-1)(t-1)+k}{t} = k-1+\frac{1}{t}.$$

Thus, $k \leq \frac{m}{t} + 1 - \frac{1}{t} < \frac{m}{t} + 1$. Hence, $AR_t(G) = \lceil \frac{m}{t} \rceil$ and so (a) holds.

On the other hand, if $m > t(t-1)$, then $k \geq t$ and Corollary 3.2 applies. We show that

$$k = \left\lceil \frac{-3+\sqrt{8m+12t-4t^2+1}}{2} \right\rceil.$$

First, since $m \leq \binom{k+2}{2} + \binom{t-1}{2} - 1$, it follows that

$$8m \leq (4k^2+12k+8) + (4t^2-12t+8) - 8.$$

Thus,

$$(2k+3)^2 = 4k^2+12k+9 \geq 8m+12t-4t^2+1$$

and so

$$2k+3 \geq \sqrt{8m+12t-4t^2+1}.$$

Here, observe that $8m+12t-4t^2+1 \geq 0$ as $m > t(t-1)$ so that we are taking the square root of a non-negative number. Hence,

$$k \geq \frac{-3+\sqrt{8m+12t-4t^2+1}}{2}.$$

Next, since $m \geq \binom{k+1}{2} + \binom{t-1}{2}$, it follows that

$$m > \binom{k+1}{2} + \binom{t-1}{2} - 1 = \frac{k^2 + k + t^2 - 3t}{2}.$$

Thus,

$$8m > 4k^2 + 4k + 4t^2 - 12t$$

and so

$$(2k+1)^2 = 4k^2 + 4k + 1 < 8m + 12t - 4t^2 + 1.$$

Therefore, $2k + 1 < \sqrt{8m + 12t - 4t^2 + 1}$ and so

$$k < \frac{-1 + \sqrt{8m + 12t - 4t^2 + 1}}{2} = \frac{-3 + \sqrt{8m + 12t - 4t^2 + 1}}{2} + 1.$$

Hence, $AR_t(G) = \left\lceil \frac{-3 + \sqrt{8m + 12t - 4t^2 + 1}}{2} \right\rceil$ and so (b) holds. $\square$

To illustrate Corollary 3.3, we determine $AR_t(G)$ where $G \in \{K_{1,20}, 20K_2\}$ for $t = 5$ and $t = 3$. Here, $m = 20$. First, let $t = 5$. Since $20 \leq 5(5-1) = 20$, it follows by Corollary 3.3(a) that $AR_5(G) = k$ where k is the unique positive integer such that $5k - 4 \leq 20 \leq 5k$. Hence, $k = \lceil \frac{20}{5} \rceil = 4$ and so $AR_5(G) = 4$. Next, let $t = 3$. Since $20 > 3(3-1) = 6$, it follows by Corollary 3.3(b) that $AR_3(G) = k$ where k is the unique positive integer such that $\binom{k+1}{2} + 1 \leq 20 \leq \binom{k+2}{2}$. Hence,

$$k = \left\lceil \frac{-3 + \sqrt{8 \cdot 20 + 12 \cdot 3 - 4 \cdot 3^2 + 1}}{2} \right\rceil = 5$$

and so $AR_3(G) = 5$.

While the primary question here is that of determining $AR_t(G)$ for a graph G of size m and a positive integer t, there are other questions of a general nature, including the following.

> *For integers t and k where $t, k \geq 2$, what is the maximum size of a graph G for which $AR_t(G) = k$?*

As an illustration of this, we answer this question for $t = 3$ when $k = 2$ and $k = 3$. By Corollaries 3.1 and 3.2, if $G = K_{1,m}$ or $G = mK_2$, then (1) $AR_3(G) = 2$ if and only if $4 \leq m \leq 6$, (2) $AR_3(G) = 3$ if and only if $7 \le m \le 10$, and (3) $AR_3(G) = 4$ if and only if $11 \le m \le 15$. This, however, is not true in general. In fact, the maximum size of a graph G for which $AR_3(G) = 2$ is 9 and the maximum size of a graph G for which $AR_3(G) = 3$ is 18. In order to verify these facts, we first present the following result.

Proposition 3.4. *Let $t \ge 2$ be an integer. If every graph G of size m has $AR_t(G) \ge k$, then every graph H of size greater than m has $AR_t(H) \ge k$.*

Proof. Let c be an arbitrary t-edge coloring of a graph H of size greater than m that results in t monochromatic (nonempty) graphs H_1, H_2, ..., H_t, where every edge of H_i is colored i for $i = 1, 2, 3, \ldots, t$. For $1 \le i \le t$, let $X_i \subseteq E(H_i)$ be a nonempty set such that $\sum_{i=1}^{t} |X_i| = m$. Let $X = \cup_{i=1}^{t} X_i$ and let $G = H[X]$ be the subgraph in H induced by X. We define a t-coloring c' of G by $c'(e) = c(e)$ for each $e \in X$. Since $AR_t(G) \ge k$ by the assumption, there is an ascending Ramsey sequence of length k in G with respect to c'. Hence, there is an ascending Ramsey sequence of length k in H with respect to c. Therefore, $AR_t(H) \ge k$. $\square$

It is well known that all graphs of small size possess an ascending subgraph decomposition (for example, all graphs of size less than 15). We will use this fact in the subsequent discussion. For two vertex-disjoint graphs F and G, let $F + G$ denote the union of F and G with $V(F + G) = V(F) \cup V(G)$ and $E(F + G) = E(F) \cup E(G)$. For an integer $n \ge 2$, let P_n denote the path of order n and size $n - 1$. For an integer $k \ge 2$, let M_k denote the maximum size of a graph G for which $AR_3(G) = k$. We are now prepared to determine M_k when $k = 2$ and $k = 3$.

Theorem 3.5. $M_2 = 9$ *and* $M_3 = 18$.

Proof. We begin by verifying that $M_2 = 9$. First, we show that there are graphs of size 9 having ascending 3-Ramsey index 2. For example, let $G = 5K_2 + K_{1,4}$ and let $c : E(G) \to [3]$ be a 3-edge coloring of G that results in three monochromatic graphs K_2, $4K_2$ and $K_{1,4}$ whose edges are colored 1, 2, 3, respectively. Then there is no ascending Ramsey sequence of length 3 in G. Thus, $AR_3(G) \le 2$ and so $AR_3(G) = 2$.

Next, we show that every graph of size 10 or more has ascending 3-Ramsey index at least 3. By Proposition 3.4, it suffices to show that every graph of size 10 has ascending 3-Ramsey index at least 3. Let H be a graph of size 10 and let c be a 3-edge coloring of H using the colors in [3]. We show that there is an ascending Ramsey sequence of length 3 in H with respect to c.

For $1 \leq i \leq 3$, let H_i be the (nonempty) subgraph of size m_i induced by the set of the edges colored i in H. We may assume that $1 \leq m_1 \leq m_2 \leq m_3$ and so $4 \leq m_3 \leq 8$. If $m_3 \geq 6$, then H_3 has an ascending subgraph decomposition. Thus, H_3 and H as well has an ascending Ramsey sequence of length 3. Thus, we may assume that $m_3 = 5$ or $m_3 = 4$.

First, suppose that $m_3 = 5$. If H_3 is a star or a matching, then H_3 can be decomposed into $K_{1,3}$ and $K_{1,2}$ or $3K_2$ and $2K_2$, respectively, and so there is an ascending Ramsey sequence of length 3 in H. If H_3 is neither a star nor a matching, then it is easy to see that H_3 can be decomposed into two subgraphs A and B where $A \in \{P_4, P_3 + K_1\}$ and $B \in \{P_3, 2K_2\}$. Since $B \subseteq A$ for each choice of A and B, there is an ascending Ramsey sequence of length 3 in H.

Next, suppose that that $m_3 = 4$. Then $(m_2, m_1) \in \{(4, 2), (3, 3)\}$. First, suppose that $H_3 = K_{1,4}$ is star. If $P_3 \subseteq H_2$ or $P_3 \subseteq H_1$, then there is an ascending Ramsey sequence of length 3 in H. Thus, we may assume that H_2 and H_1 are both matchings. Then either $(H_2, H_1) = (4K_2, 2K_2)$ or $H_2 = H_1 = 3K_2$. In either case, there is also an ascending Ramsey sequence of length 3 in H. Next, suppose that $H_3 = 4K_2$ is a matching. If $2K_2 \subseteq H_2$ or $2K_2 \subseteq H_1$, then there is an ascending Ramsey sequence of length 3 in H. Thus, we may assume that H_2 and H_1 are (i) both stars, (ii) both are K_3, or (iii) one is a star and the other is K_3. In any case, the subgraph $H[E(H_1) \cup E(H_2)]$ induced by $E(H_1) \cup E(H_2)$ contains edge-disjoint copies of either a monochromatic $K_{1,3}$ or K_3 and a monochromatic P_3, producing an ascending Ramsey sequence of length 3 in H. Finally, suppose that H_3 is neither a star nor a matching. Thus, either $P_4 \subseteq H_3$ or $P_3 + K_2 \subseteq H_3$. Since $m_2 \geq 3$, it follows that H_2 contains either P_3 or $2K_2$. In either case, there is an ascending Ramsey sequence of length 3 in H.

Therefore, $AR_3(G) \geq 3$ for every graph G of size greater than 9 and so $M_2 = 9$.

Next, we verify that $M_3 = 18$. First, we show that there are graphs of size 18 having ascending 3-Ramsey index 3. For example, let $G = 2K_3 + 6K_2 + K_{1,6}$ of size 18 and let $c : E(G) \to [3]$ be a 3-edge coloring of G that results in three monochromatic graphs $2K_3$, $6K_2$, and $K_{1,6}$, whose edges are colored 1, 2, or 3, respectively. Since there is no ascending Ramsey sequence of length 4 in G, it follows that $AR_3(G) \leq 3$. It can be shown that for every 3-edge coloring c of G, there is an ascending Ramsey sequence of length 3 with respect to c. Therefore, $AR_3(G) = 3$ for $G = 2K_3 + 6K_2 + K_{1,6}$.

Next, we show that every graph of size 19 or more has ascending 3-Ramsey index at least 4. By Proposition 3.4, we show that every graph of size 19 has ascending 3-Ramsey index at least 4. Let G be a graph of size 19 and let c be a 3-edge coloring of G using the colors in [3]. We show that there is an ascending Ramsey sequence of length 4 in G with respect to c. For $1 \leq i \leq 3$, let H_i be the (nonempty) subgraph of size m_i induced by the set of the edges colored i in G. We may assume that $1 \leq m_1 \leq m_2 \leq m_3$ and so $7 \leq m_3 \leq 17$. If $m_3 \geq 10$, then H_3 has an ascending subgraph decomposition and so there is an ascending Ramsey sequence of length 4 in G. Thus, we may assume that $7 \leq m_3 \leq 9$ and so $m_2 + m_1 \geq 10$. Hence, either $m_2 \geq 7$ or $(m_2, m_1) \in \{(6,4),(5,5)\}$. Let $H \subseteq H_3$ be a monochromatic subgraph of size 7 in G. Using a case-by-case analysis, the following statement can be verified.

- $\star$ The graph H can be decomposed into two subgraphs F and F' where $|E(F)| = 4$ and $|E(F')| = 3$ such that $F' \subseteq F$ and $F' \neq K_3$.

With the aid of the statement in $(\star)$, we show next that there is an ascending Ramsey sequence s_c of length 4 in G. Suppose that H is decomposed into two graphs F and F' where $|E(F)| = 4$ and $|E(F')| = 3$ such that $F' \subseteq F$ and $F' \neq K_3$. Hence, $F' \in \{K_{1,3}, P_4, P_3 + K_2, 3K_2\}$. First, suppose that $F' = K_{1,3}$. If there is a monochromatic P_3 in $G - E(H)$, then $s_c : K_2, P_3, F', F$. If there is no a monochromatic P_3 in $G - E(H)$, then $H_1 = m_1 K_2$ and $H_2 = m_2 K_2$. Since $m_2 \geq m_1 \geq 4$, it follows that $s_c : K_2, 2K_2, 3K_3, 4K_2$. Next, suppose that $F' \in \{P_4, P_3 + K_2\}$. Since $m_2 \geq 5$, it follows that $2K_2 \subseteq H_2$. Because $2K_2 \subseteq P_4$ and $2K_2 \subseteq P_3 + K_2$, it follows that $s_c : K_2, 2K_2, F', F$. Finally, suppose that $F' = 3K_2$. If there is a monochromatic $2K_2$ in $G - E(H)$, then $s_c : K_2, 2K_2, F', F$. If there is no a monochromatic $2K_2$ in $G - E(H)$, then $H_1 = K_{1,m_1}$ and $H_2 = K_{1,m_2}$. Hence, $s_c : K_2, P_3, K_{1,3}, K_{1,4}$ is an ascending Ramsey sequence in G.

Therefore, $AR_3(G) \geq 4$ for every graph G of size greater than 18 and so $M_3 = 18$. □

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