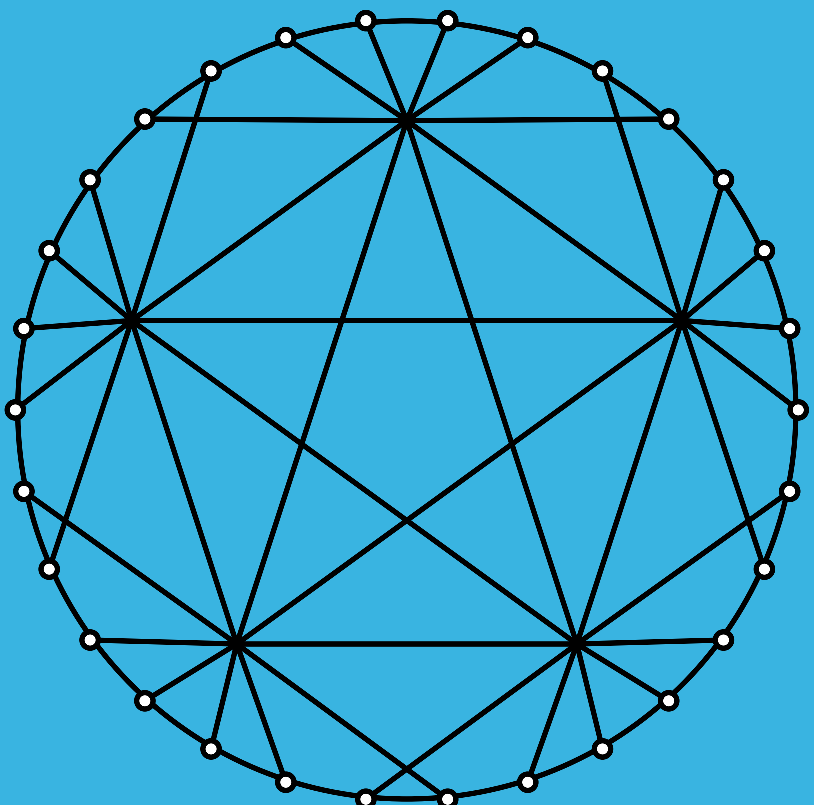
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 95
June 2022



Duluth, Minnesota, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **95** (2022), 57–69

Edge odd graceful labelings of certain new classes of graphs

K.M. Kathiresan[1], S. Muthumari[1] and R. Ramalakshmi*[2]

[1]Centre for Graph Theory, Ayya Nadar Janaki Ammal College (Autonomous)Sivakasi-626 124, India.
kathir2esan@yahoo.com, smuthumari276@gmail.com
[2]Department of Mathematics, Rajapalayam Rajus' College, Rajapalayam-626 117, India.
ramimani20@gmail.com

Abstract. In this paper we introduce three families of graphs and we discuss the existence or non existence of edge odd graceful labeling for these classes of graphs.

1 Introduction

Solairaju and Chitra [10] defined a graph G with q edges to be edge odd graceful if there is a bijection f from the edges of the graph to

$$\{1, 3, 5, \ldots, 2q-1\}$$

such that, when each vertex is assigned the sum of all the edges incident to it modulo $2q$, the resulting vertex labels are distinct. If G has n vertices, the corona of G with H, $G \odot H$ is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H. They [10] proved that the following graphs are odd graceful: path with at least 3 vertices; odd cycles; ladders $P_n \times P_2$ $(n \geq 3)$; stars with an even number of edges; and crowns $C_n \odot K_1$.

*Corresponding author.
Key words and phrases: Edge odd graceful graphs, Join Path
AMS (MOS) Subject Classifications: 05C78

Received: 24 January 2022
Accepted: 16 March 2022

In [9] they proved that the following graphs have edge odd graceful labeling: $P_n(n > 1)$ with a pendant edge attached to each vertex (combs); the graph obtained by appending $2n + 1$ pendant edges to each endpoint of P_2 or P_3; and the graph obtained by subdividing each edge of the star $K_{1,2n}$.

Singhun [8] proved that the following graphs have edge odd graceful labelings: W_n , $W_{2n} \odot K_1$ and $W_n \odot K_m$ when n is odd, m is even, and n divides m.

Seoud and Salim [7] presented an edge odd graceful labelings for the following families of graphs : W_n for $n \equiv 1, 2$ and $3 \pmod 4$, $C_n \odot \overline{K}_{2m-1}$, even helms $P_n \odot K_{2m}$ and $K_{2,s}$. They proved that the trees of odd number of vertices and odd degrees can't be edge odd graceful graphs. Also they proved that the cycle C_n is not an edge odd graceful graph when n is even. Finally they produced a simple way to label complete graphs which provides edge odd graceful labeling to a good number of complete graphs K_n within the range $n \in \{4, 5, ..., 99, 100\}$. In 2007, Gao [4] proved the existence of odd graceful labeling of some union of graphs. In 2019, Daoud [3] proved the necessary and sufficient conditions for the Cylinder grid graph $C_{m,n} = P_m \times C_n$ and torus grid graph $T_{m,n} = C_m \times C_n$ to be edge odd graceful. In this paper we discuss the Edge odd graceful labeling of some new classes of graphs namely, $K_n^c \vee 2K_2$, $P_{a,b}$ and Flower graph FL_n.

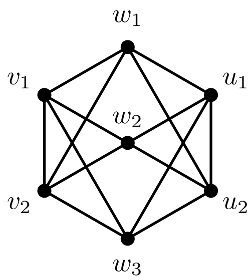
2 Edge odd gracefulness of $K_n^c \vee 2K_2$

The graph $K_n^c \vee 2K_2$ is the join of the complement of the complete graph on n vertices and two disjoint copies of K_2. Rao-Hebbare [6] conjectured that for each positive integer n, the graph $K_n^c \vee 2K_2$ is not graceful. This was proved by Bhat-Nayak and Gokhale [2]. In [1] Balakrishnan and Sampathkumar proved that $K_n^c \vee 2K_2$ is magic under the conditions that $n = 3$ and also they proved that for any positive integer n, the graph $K_n^c \vee 2K_2$ is antimagic and this graph is harmonious if and only if n is even.

In this paper we prove that $K_n^c \vee 2K_2$ is edge odd graceful for all positive integers n. Throughout this paper, we denote the set of vertices of $K_n^c \vee 2K_2$ by $\{v_1, v_2, u_1, u_2, w_1, w_2, \ldots, w_n\}$ so that its edge set is $\{v_1v_2, u_1u_2\} \cup \{u_1w_i, u_2w_i, v_1w_i, v_2w_i | 1 \leq i \leq n\}$.

Definition 2.1. *The join $K_n^c \vee 2K_2$ is the graph obtained by taking a copy of K_n^c and two adjacent copies of K_2 disjoint from K_n^c and joining every vertex of K_n^c to every vertex of $2K_2$.*

Example 2.2. *The graph of $K_3^c \vee 2K_2$.*



Theorem 2.3. *$G = K_n^c \vee 2K_2$ is an edge odd graceful graph for all positive integers n.*

Proof. Let $V(K_n^c \vee 2K_2) = \{v_1, v_2, u_1, u_2, w_1, w_2, \ldots, w_n\}$ and $E(K_n^c \vee 2K_2) = \{v_1v_2, u_1u_2, u_1w_k, u_2w_k, v_1w_k, v_2w_k | 1 \leq k \leq n\}$ as shown in Figure 1. Therefore, $p = |V(G)| = n + 4$ and $q = |E(G)| = 4n + 2$.

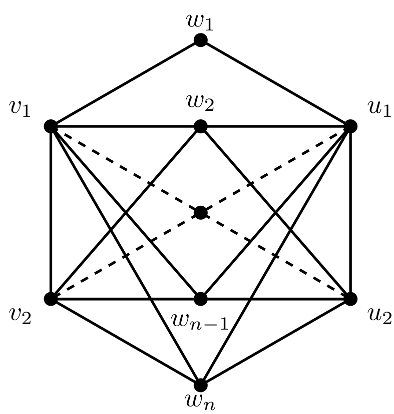
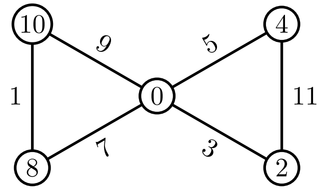


Figure 1: The graph of $K_n^c \vee 2K_2$.

Case (i): When $n = 1$.



The above labeling shows that $K_1^c \vee 2K_2$ is an edge odd graceful graph.

Case (ii): When $n > 1$.

Define a mapping f from $E(G)$ to $\{1, 3, \ldots, 2q-1\}$ by,

$$\begin{aligned} f(u_1u_2) &= 1; \\ f(v_1v_2) &= 3; \\ f(v_1w_k) &= 2k+3; 1 \leq k \leq n \\ f(v_2w_k) &= 2n+2k+3; 1 \leq k \leq n \\ f(u_1w_k) &= 4n+2k+3; 1 \leq k \leq n \\ f(u_2w_k) &= 6n+2k+3; 1 \leq k \leq n \end{aligned}$$

The induced mapping is given by,

$$\begin{aligned} f^*(v_1) &= f(v_1v_2) + \sum_{k=1}^{n} f(v_1w_k) \pmod{8n+4} \\ &= 3 + \sum_{k=1}^{n} [2k+3] \pmod{8n+4} \\ &= n^2 + 4n + 3 \pmod{8n+4} \end{aligned}$$

$$\begin{aligned} f^*(v_2) &= f(v_1v_2) + \sum_{k=1}^{n} f(v_2w_k) \pmod{8n+4} \\ &= 3 + \sum_{k=1}^{n} [2n+2k+3] \pmod{8n+4} \\ &= 3n^2 + 4n + 3 \pmod{8n+4} \end{aligned}$$

$$\begin{aligned} f^*(u_1) &= f(u_1u_2) + \sum_{k=1}^{n} f(u_1w_k) \pmod{8n+4} \\ &= 1 + \sum_{k=1}^{n} [4n+2k+3] \pmod{8n+4} \\ &= 5n^2+4n+1 \pmod{8n+4} \end{aligned}$$

$$\begin{aligned} f^*(u_2) &= f(u_1u_2) + \sum_{k=1}^{n} f(u_2w_k) \pmod{8n+4} \\ &= 1 + \sum_{k=1}^{n} [6n+2k+3] \pmod{8n+4} \\ &= 7n^2+4n+1 \pmod{8n+4} \end{aligned}$$

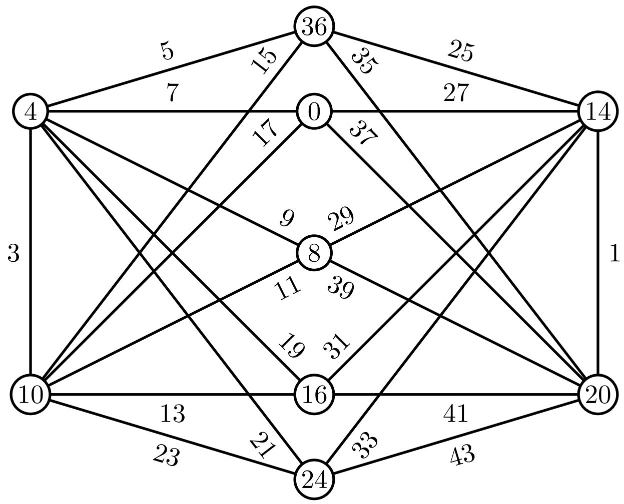
$$\begin{aligned} f^*(w_k) &= f(u_1w_k) + f(u_2w_k) + f(v_1w_k) + f(v_2w_k) \pmod{8n+4} \\ &= (4n+2k+3) + (6n+2k+3) + (2k+3) \\ &\quad + (2n+2k+3) \pmod{8n+4} \\ &= 12n+8k+12 \pmod{8n+4} \\ &= 4n+8k+8 \pmod{8n+4}; 1 \le k \le n. \end{aligned}$$

The labels of edges are in the set $\{1\} \cup \{3\} \cup \{5, 7, \ldots, 2n+3\} \cup \{2n+5, \ldots, 4n+3\} \cup \{4n+5, \ldots, 6n+3\} \cup \{6n+5, \ldots 8n+3\}$.

Then the labels of vertices are in the set $\{n^2+4n+3\} \cup \{3n^2+4n+3\} \cup \{5n^2+4n+1\} \cup \{7n^2+4n+1\} \cup \{4n+8i+8 | 1 \le i \le n\} \pmod{8n+4}$.
That is, $\{n^2+4n+3\} \cup \{3n^2+4n+3\} \cup \{5n^2+4n+1\} \cup \{7n^2+4n+1\} \cup \{4n+16, 4n+24, \ldots, 4n+8n, 4n+4\} \pmod{8n+4}$.

We observe that the vertices have distinct labels. Therefore, $K_n^c \vee 2K_2$, is an edge odd graceful graph. $\square$

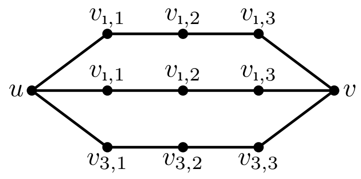
Example 2.4. *An edge odd graceful labeling of* $K_5^c \vee 2K_2$.



3 Edge odd gracefulness of $P_{a,b}$

Let u and v be two fixed vertices. We connect u and v by means of $b \geq 2$ internally disjoint paths of length $a \geq 2$ each. Kathiresan [4] proved that $P_{2r,2m+1}$ are graceful for all values of r and m. He also conjectured that $P_{a,b}$ is graceful except when $a = 2r + 1$ and $b + 4s + 2$. Throughout this paper we denote the set of vertices of $P_{a,b}$ by $\{u, v, v_{j,i} | 1 \leq i \leq a-1, 1 \leq j \leq b\}$, so that its edge set is $\{uv_{j,i}, v_{j,a-1}v, v_{j,i}v_{j,i+1} | 1 \leq i \leq a-1, 1 \leq j \leq b\}$.

Example 3.1. *The graph* $P_{4,3}$



Theorem 3.2. $G = P_{a,b}$ *is an edge odd graceful graph when both* a *and* b *are odd.*

Proof. Let $V(G) = \{u, v, v_{j,i} | 1 \leq i \leq a-1, 1 \leq j \leq b\}$, and $E(G) = \{uv_{j,i}, v_{j,a-1}v, v_{j,i}v_{j,i+1} | 1 \leq i \leq a-1, 1 \leq j \leq b\}$ as shown in Figure 2. Therefore $p = |V(G)| = ab - b + 2$ and $q = |E(G)| = ab$ where $u = v_{1,0} = v_{2,0} = \cdots = v_{b,0}$ and $v = v_{1,a} = v_{2,a} = \cdots = v_{b,a}$. Define a labeling f from

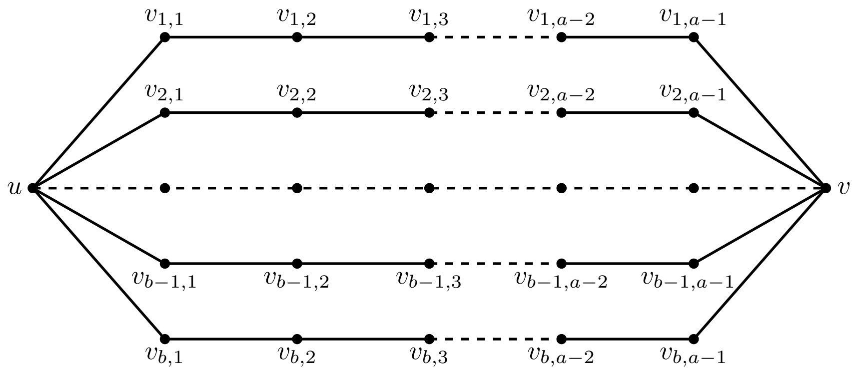


Figure 2: The graph of $P_{a,b}$.

$E(G)$ to $\{1, 3, \ldots, 2q-1\}$ by,

$$f(v_{1,i}v_{1,i+1}) = 2i + 1; 0 \leq i \leq a-1$$
$$f(v_{j,i}v_{j,i+1}) = 2ja + 2i + 1; 0 \leq i \leq a-1, 2 \leq j \leq b-1$$
$$f(v_{b,i}v_{b,i+1}) = 4a - 2i - 1; 0 \leq i \leq a-1.$$

The induced mapping is given by,

$$\begin{aligned} f^*(u) &= f(uv_{1,1}) + f(uv_{b,1}) + f(uv_{j,1}) \pmod{2ab} \\ &= 1 + (4a-1) + \sum_{j=2}^{b-1}[2ja+1] \pmod{2ab} \\ &= 1 + 4a - 1 + (b-2) + ab(b-1) - 2a \pmod{2ab} \\ &= ab^2 - ab + 2a + b - 2 \pmod{2ab} \end{aligned}$$

$$f^*(v) = f(vv_{1,b-1}) + f(vv_{b,a-1}) + \sum_{i=2}^{b-1} f(vv_{j,a-1}) \pmod{2ab}$$
$$= (2a+1) + (4a-2a-1) + \sum_{j=2}^{b-1}[2ja+2a-1] \pmod{2ab}$$
$$= 4a-(b-2)+2a(b-2)+ab(b-1)-2a \pmod{2ab}$$
$$= 4a-b+2+2ab-4a+ab^2-ab-2a \pmod{2ab}$$
$$= ab^2+ab-2a-b+2 \pmod{2ab}$$

$$f^*(v_{j,i}) = f(v_{j,i-1}v_{j,i}) + f(v_{j,i}v_{j,i-1}) \pmod{2ab}$$
$$= 2ja+2(i-1)+1+2ja+2i+1 \pmod{2ab}$$
$$= 4ja+4i \pmod{2ab}; 1 \le i \le a-1; 2 \le j \le b-1;$$

$$f^*(v_{1,i}) = f(v_{1,i-1}v_{1,i}) + f(v_{1,i}v_{1,i+1}) \pmod{2ab}$$
$$= 2(i-1)+1+2i+1 \pmod{2ab}$$
$$= 4i \pmod{2ab}; 1 \le i \le a-1$$

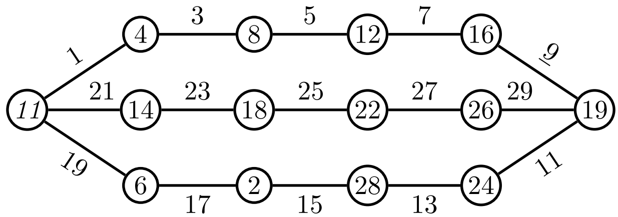
$$f^*(v_{b,i}) = f(v_{b,i-1}v_{b,i}) + f(v_{b,i}v_{b,i+1}) \pmod{2ab}$$
$$= 4a-2(i-1)-1+4a-2i-1 \pmod{2ab}$$
$$= 8a-4i \pmod{2ab}; 1 \le i \le a-1$$

The labels of edges are in the set $\{1,3,\ldots,2a-1\} \cup \{2a+1, 2a+3, \ldots, 4a-1\} \cup \{4a+1, 4a+3, \ldots, 2ab-1\}$.

Then the labels of vertices are in the set $\{ab^2-ab+2a+b-2\} \cup \{ab^2+ab-2a-b+2\} \cup \{4ja+4i | 1 \le i \le a-1; 2 \le j \le b-1\} \cup \{4i | 1 \le i \le a-1\} \cup \{8a-4i | 1 \le i \le a-1\} \pmod{2ab}$.

We observe that the vertices have distinct labels. Therefore $P_{a,b}$ is an edge odd graceful graph. □

Example 3.3. *An edge odd graceful labeling of $P_{5,3}$.*



Now, we discuss the non existence of edge odd graceful labeling of the graph $P_{a,2}$.

Theorem 3.4. *The graph $P_{a,2}$ is not an edge odd graceful graph when $a \geq 2$.*

Proof. Suppose that, $P_{a,2}$ admits an edge odd graceful labeling, then the labels of edges are in the set: $\{1, 3, 5, \ldots, 2q-1\}$. we can get the labels of vertices as $0, 2, 4, \ldots, 2q-2 \pmod{2q}$. Then

(i) $\sum_{v \in V(P_{a,2})} f^*(v) = 2 \sum_{v \in V(P_{a,2})} f(e) \equiv 2q^2 \equiv 0 \pmod{2q}$.

(ii) $\sum_{v \in V(P_{a,2})} f^*(v) = \sum_{i=0, i-even}^{2q-2} i \equiv q(q-1) \equiv q \pmod{2q}$.

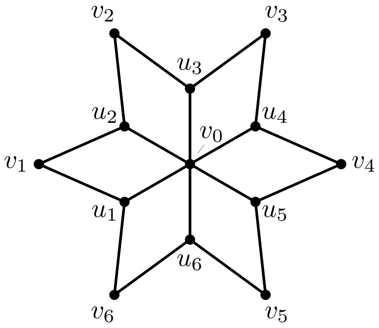
The difference in the above (i) and (ii) leads to a contradiction. Therefore $P_{a,2}$ is not an edge odd graceful graph. □

We conclude this section with the subsequent open problem: Discuss the existence or non existence $P_{a,2}$ for the remaining cases.

4 Edge odd gracefulness of FL_n

Definition 4.1. *The Flower graph, FL_n , is the graph with $V(FL_n) = \{u_k, v_0, v_k | 1 \leq k \leq n\}$; and $E(FL_n) = \{v_0u_k, v_ku_k | 1 \leq k \leq n\} \cup \{v_ku_{k+1} | 1 \leq k \leq n-1\} \cup \{v_nu_1\}$.*

Example 4.2. *The graph of FL_6.*



Theorem 4.3. *The Flower graph $G = FL_n$ is an edge odd graceful graph when $n \geq 3$.*

Proof. Let $V(G) = \{u_k, v_0, v_k | 1 \leq k \leq n\}$, $E(G) = \{v_0u_k, v_ku_k | 1 \leq k \leq n\} \cup \{v_ku_{k+1} | 1 \leq k \leq n-1\} \cup \{v_nu_1\}$ as shown in Figure 3. Therefore, $p = |V(G)| = 2n + 1$ and $q = |E(G)| = 3n$.

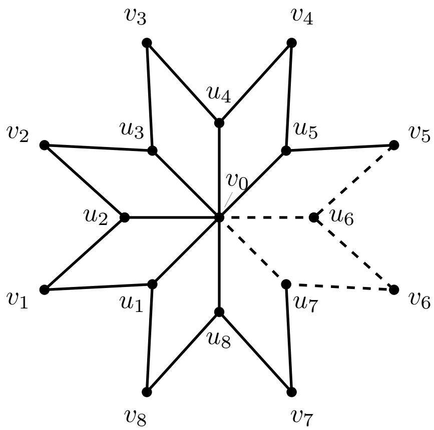


Figure 3: The graph of FL_n.

Case (i): When $n = 3$.

The labeling of the graph given in Figure 4 shows that, FL_3 is an edge odd graceful graph.

Case (ii): When $n > 3$.

Define a labeling f from $E(G)$ to the set $\{1, 3, 5, \ldots, 2q - 1\}$ by,

$$f(v_ku_k) = 2k - 1; 1 \leq k \leq n$$
$$f(v_ku_{k+1}) = 2n + 2k - 1; 1 \leq k \leq n - 1$$
$$f(v_0u_1) = 4n + 1;$$
$$f(v_0u_k) = 6n - 2k + 3; 2 \leq k \leq n$$

The induced mapping is given by,

$$\begin{aligned} f^*(v_k) &= f(v_ku_k) + f(v_ku_{k+1}) \pmod{6n} \\ &= (2k - 1) + (2n + 2k - 1) \pmod{6n} \\ &= 2n + 4k - 2 \pmod{6n}; 1 \leq k \leq n \end{aligned}$$

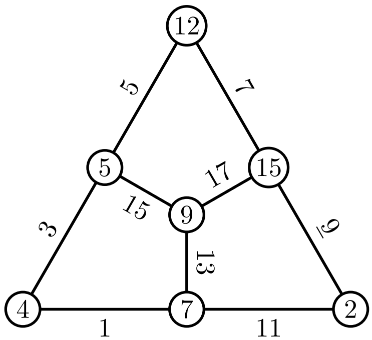


Figure 4: An edge odd graceful labeling of FL_3.

$$\begin{aligned} f^*(u_1) &= f(v_1u_1) + f(v_0u_1) + f(u_1v_n) \pmod{6n} \\ &= 1 + (4n+1) + (4n-1) \pmod{6n} \\ &= 8n+1 \pmod{6n} = 2n+1 \pmod{6n} \end{aligned}$$

$$\begin{aligned} f^*(u_k) &= f(v_ku_k) + f(v_0u_k) + f(v_ku_{k+1}) \pmod{6n} \\ &= (2k-1) + (6n-2k+3) + (2n+2(k-1)-1) \pmod{6n} \\ &= 2n+2k-1 \pmod{6n}; 2 \le k \le n \end{aligned}$$

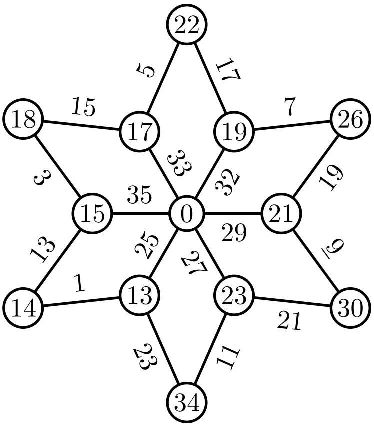
$$\begin{aligned} f^*(v_0) &= f(v_0u_1) + \sum_{k=2}^{n} f(v_0u_k) \pmod{6n} \\ &= 4n+1 + \sum_{k=2}^{n}[6n-2+3] \pmod{6n} \\ &= 4n+1+6n(n-1)+3(n-1)-n(n-1) \pmod{6n} \\ &= 5n^2 \pmod{6n} \end{aligned}$$

The labels of the edges are in the set $\{1,3,\ldots,2n-1\} \cup \{2n+1, 2n+3, \ldots, 4n-1\} \cup \{4n+1\} \cup \{4n+3, 4n+5, \ldots, 6n-1\}$. Then the labels of vertices in the set $\{2n+2, 2n+6, ..., 6n-2\} \cup \{2n+1\} \cup \{2n+3, 2n+5, \ldots, 4n-1\} \cup \{5n^2\} \pmod{6n}$.

We observe that, the vertices have distinct labels.

Therefore FL_n is an edge odd graceful graph. □

Example 4.4. *An edge odd graceful labeling of $FL-6$.*



Acknowledgement:

The authors would like to thank the referees for their useful suggestions and comments.

References

[1] R. Balakrishnan and R. Sampahkumar, Decompositions of regular graphs into $K_n^c \vee 2K_2$, *Discrete Math.*, **156** (1996), 19–28.

[2] V.N. Bhat-Nayak and S.K. Gokhale, Validity of Hebbare's conjecture, *Util. Math.*, **29** (1986), 49–59.

[3] S.N. Daoud, Edge odd graceful labeling of Cylinder and Torus Grid graphs, *IEEE Access*, **7** (2019), 1–25.

[4] Z. Gao, Odd graceful labelings of some union graphs, J.Nat. *Sci. Heilongjiang Univ.*, **24** (2007), 35–39.

[5] K.M. Kathiresan, Two classes of graceful graphs, *Ars Combin.*, **55** (2000), 129–132.

[6] S.P. Rao-Hebbare, Graceful cycles, *Util. Math.*, **10** (1976), 307–317.

[7] M.A. Seoud and M.A. Salim, Further results on edge odd graceful graphs, *Turkish J. Math.*, **40** (2016) 3, 647–656.

[8] S. Singhun, Graph with edge odd graceful labeling, *Math. Forum*, **8** (2013) 577–582.

[9] A. Solairaju and K. Chitra, Edge odd graceful labeling of some graphs, *Proceedings of ICMCS*, **1** (2008), 101–107.

[10] A. Solairaju and K. Chitra, Edge odd graceful graphs, *Electron. Notes Discrete Math.* **33** (2009),15–20.