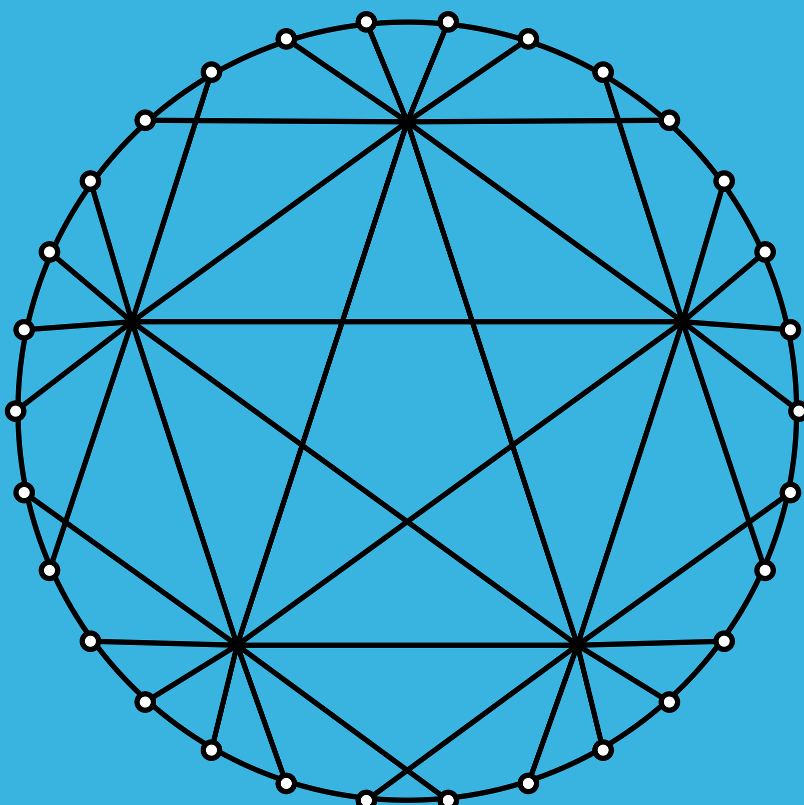
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 98
June 2023



Duluth, Minnesota, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **98** (2023), 43–59

A chessboard problem and irregular domination

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In memory of Henda Swart

Abstract. A chessboard problem involving chess pieces called princes is described which leads to a study of a topic called irregular domination of graphs. A connected graph has an irregular dominating set S if the vertices of S can be labeled with distinct positive integers in such a way that for every vertex v of G, there is a vertex $u \in S$ such that the distance from v to u is the label assigned to u. It is shown that for infinite classes of grids studied, all have irregular dominating sets with small exceptions.

1 Introduction

There is a familiar problem involving the chess piece queen (see [4, 9], for example). A single move by a queen on a chessboard consists of moving the queen along any number of vacant squares diagonally, horizontally, or vertically. This problem is sometimes referred to as

> **The five queens problem**
>
> *Can five queens be placed on distinct squares of the standard 8×8 chessboard in such a way that every vacant square can be attacked by at least one of these queens?*

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Key words and phrases: chessboard, distance, vertex orbits, domination, irregular domination.
AMS (MOS) Subject Classifications: 05C05, 05C12, 05C15, 05C69.

Received: 2 December 2021
Accepted: 26 January 2022

It is well known that the answer to this question is yes and that this cannot be done with four queens. In fact, five queens can be so located on a chessboard such that either (a) each queen can be attacked by another queen or (b) no queen can be attacked by another queen.

We now describe another problem involving a chessboard, but this involves a different kind of chess piece that moves in a different way on a standard 8×8 chessboard. We refer to such a chess piece as a *prince*. For an integer k with $1 \leq k \leq 14$, a *k-prince* is a chess piece that is permitted to move horizontally and/or vertically a total of exactly k squares (vacant or not) *away from* its current position. The k-prince is then said to have *covered* or *attacked* the resulting square to which the k-prince has moved. A 14-prince can only attack one square and this can occur only if the 14-prince is located on one of the four corner squares of the chessboard, in which case the 14-prince attacks only the opposite corner square. A 3-prince, if properly positioned on a chessboard, can move three squares horizontally or vertically, as a rook can do, or move two squares horizontally or vertically followed by one square in a perpendicular direction (equivalently one square horizontally or vertically followed by two squares in a perpendicular direction), as a knight can do. Figure 1 shows the twelve squares (marked $\star$) that a 3-prince $3P$ can attack if $3P$ is placed on one of the four most central squares of the chessboard.

			$\star$				
		$\star$		$\star$			
	$\star$				$\star$		
$\star$			$3P$			$\star$	
	$\star$				$\star$		
		$\star$		$\star$			
			$\star$				

Figure 1: Squares attacked by a 3-prince on a standard 8×8 chessboard

This leads us to another chessboard problem.

The fourteen princes problem

Can the fourteen k-princes $(k = 1, 2, 3, \ldots, 14)$ be placed on fourteen distinct squares of a standard 8×8 chessboard in such a way that every square (including each occupied square) can be attacked by at least one prince?

The answer to the fourteen princes problem is yes and a solution is shown in Figure 2 where the location of a k-prince on a square is indicated by labeling the square by k.

				10			14
							8
9	5	3	1	4	2	7	6
11							
	13						12

Figure 2: A solution to the fourteen princes problem

The fourteen princes problem is closely related to the topic of domination in graphs. In recent decades, domination in graphs has become a popular area of study. While this area evidently began with the work of Berge [2] in 1958 and Ore [11] in 1962, domination did not become an active area of research until 1977 with the appearance of a survey paper by Cockayne and Hedetniemi [7]. Since then, a number of variations and applications of domination have surfaced (see [9]). For a vertex v in a nontrivial connected graph G, let $N(v)$ denote the *neighborhood* of v and $N[v] = \{v\} \cup N(v)$ the *closed neighborhood* of v. A vertex v in a graph G is said to *dominate* a vertex u if either $u = v$ or $uv \in E(G)$. That is, a vertex v dominates the vertices in its closed neighborhood $N[v]$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by at least one vertex in S. The minimum number of vertices in a dominating set of G is the *domination number* $\gamma(G)$ of G.

Of the many variations of domination that have been introduced, probably the most common and most studied is total domination, introduced by Cockayne, Dawes, and Hedetniemi [6]. In total domination, a vertex u (totally) dominates a vertex v in a graph G if uv is an edge of G and so a vertex does not dominate itself. It is this type of domination that we use here, that is, in this paper, domination is total domination. A set S of vertices in a graph G is a *total dominating set* of G if for every vertex v of G, there is a vertex $u \in S$ such that u dominates v. The minimum cardinality of a total dominating set of G is the *total domination number* $\gamma_t(G)$ of G. A graph G has a total dominating set if and only if G has no isolated vertices. The book by Henning and Yeo [10] deals exclusively with total domination in graphs.

Total domination, as well as other types of domination, can be described with the aid of distance in graphs. We denote the *distance* (the length of a shortest path) between two vertices u and v in a graph G by $d(u, v)$. The greatest distance from a vertex v to a vertex of G is its *eccentricity*, denoted by $e(v)$. The minimum eccentricity among the vertices of G is the *radius* $\operatorname{rad}(G)$ of G and the maximum eccentricity is the *diameter* $\operatorname{diam}(G)$. Therefore, the diameter of G is the maximum distance between any two vertices of G. In total domination, a vertex u dominates a vertex v if $d(u, v) = 1$. For a total dominating set S in a nontrivial connected graph G, one can think of assigning each vertex of S the label 1 and assigning no label to the vertices of G not in S. Thus, if $u \in S$, then u is labeled 1, indicating that u dominates all vertices of G whose distance from u is 1. Thus, every vertex of G has distance 1 from at least one vertex of S.

In [8], a generalization of (total) domination was introduced called *orbital domination*. For a positive integer r and a vertex v in a connected graph G, the *r-orbit* $O_r(v)$ of v is $O_r(v) = \{u \in V(G) : d(u, v) = r\}$. A set $S = \{u_1, u_2, \ldots, u_k\}$ of vertices in a nontrivial connected graph G is an *orbital dominating set* of G if each vertex $u_i \in S$ can be labeled with a positive integer r_i, where $r_i \le e(u_i)$, such that $\bigcup_{i=1}^{k} O_{r_i}(u_i) = V(G)$. Thus, if S is an orbital dominating set of G, then for every vertex v of G, there exists a vertex u_i in S such that $d(u_i, v) = r_i$. Here, u_i is said to *dominate* v. The minimum cardinality of an orbital dominating set is called the *orbital domination number* of G. This concept has been studied further in [5].

If all labels of an orbital dominating set S are the same positive integer r, then S is an *r-regular orbital dominating set*. It was shown in [8] that a nontrivial connected graph G has an r-regular orbital dominating set if and only if $1 \le r \le \operatorname{rad}(G)$. If $r = 1$, then S is a *total dominating set*.

In the book [1] various "regularity" concepts are discussed, describing how this can lead to concepts that are in a sense opposite to these, resulting in "irregularity" concepts. In terms of domination, if no two vertices of an orbital dominating set S have the same label, then S is an *irregular orbital dominating set* or, more simply, an *irregular dominating set*. Consequently, a connected graph G has an irregular dominating set if it is possible to assign distinct labels (positive integers) to some vertices of G in such a way that for every vertex v of G, there is a labeled vertex u such that $d(u, v)$ is the label assigned to u. Such a labeling is called an *irregular dominating labeling* and u is said to *dominate* v. While every nontrivial connected graph has an orbital dominating set (indeed, a total dominating set), not every graph has an irregular dominating set.

Proposition 1.1. *No connected vertex-transitive graph has an irregular dominating set.*

Proof. Let G be a connected vertex-transitive graph of order n and diameter d and let $x \in V(G)$. For $1 \leq k \leq d$, let $|O_k(x)| = n_k$. Because $\bigcup_{k=1}^{d} O_k(x) = V(G - x)$, it follows that $\sum_{k=1}^{d} n_k = n - 1$. Since G is vertex-transitive, $|O_k(u)| = |O_k(v)| = n_k$ for every two vertices u and v of G and every integer k with $1 \leq k \leq d$. Hence, any vertex of G labeled k dominates exactly n_k vertices of G. Since $\sum_{k=1}^{d} n_k = n - 1$, at least one vertex of G is not dominated by any labeled vertex of G. $\square$

2 Irregular domination in grids

The fourteen princes problem can therefore be looked at as a problem in graph theory. A standard 8×8 chessboard can be represented by a graph whose 64 vertices are the squares of the chessboard and where two vertices are adjacent if the corresponding squares have a side in common. The resulting graph is the Cartesian product $P_8 \,\square\, P_8$ of the path P_8 of order 8 with itself. The fourteen Princes problem then becomes

Is there an irregular dominating labeling of the graph $G = P_8 \,\square\, P_8$?

The solution to the fourteen princes problem given in Figure 2 is the irregular dominating labeling of $G = P_8 \,\square\, P_8$ shown in Figure 3.

The graphs of the type $P_m \,\square\, P_n$ are called *$m \times n$ grids*. This brings up a question:

> For which pairs m, n of positive integers, is there an irregular dominating labeling of $P_m \,\square\, P_n$ (in addition to $m = n = 8$)? That is, for which pairs m, n of positive integers, is there a solution to the $m+n-2$ princes problem on the $m \times n$ chessboard?

In [3], it was shown that every path P_n has an irregular dominating labeling for all $n \geq 4$ except $n = 6$. By Proposition 1.1, there is no an irregular dominating labeling of $P_2 \,\square\, P_2$. For $n = 3, 5, 6, 7$, the grid $P_n \,\square\, P_2$ has an irregular dominating labeling, as shown in Figure 4. However, the grid $P_4 \,\square\, P_2$ does not possess such a labeling.

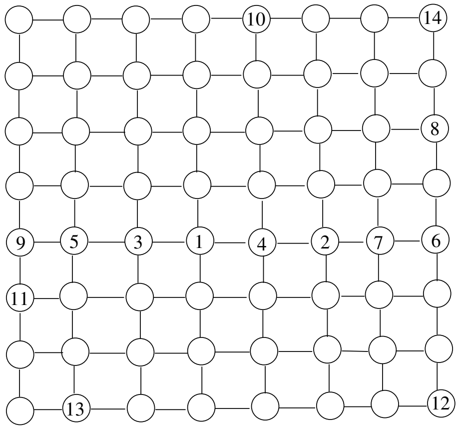

Figure 3: A labeling of $P_8 \,\Box\, P_8$

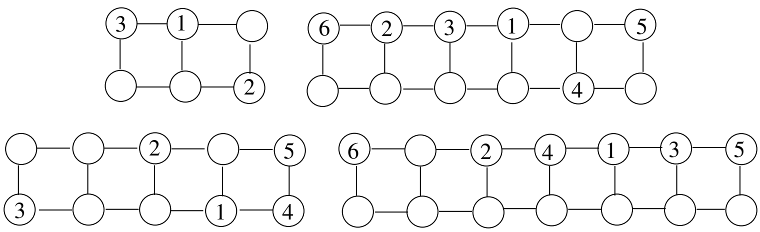

Figure 4: Irregular dominating labelings of $P_n \,\Box\, P_2$ for $n = 3, 5, 6, 7$

Proposition 2.1. *The grid $P_4 \,\Box\, P_2$ has no irregular dominating labeling.*

Proof. Let $G = P_4 \,\Box\, P_2$ be the graph consisting of two disjoint paths (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) together with the edges x_iy_i for $1 \leq i \leq 4$. The graph G is a bipartite graph of diameter 4 with partite sets $U = \{x_1, x_3, y_2, y_4\}$ and $W = \{x_2, x_4, y_1, y_3\}$. Assume, to the contrary, that there exists an irregular dominating labeling of G in which four (or fewer) vertices of G are assigned labels from the set $[4] = \{1, 2, 3, 4\}$. Observe that any vertex of G assigned an even-numbered label dominates only vertices in the partite set to which it belongs, while any vertex of G assigned an odd-numbered label dominates only vertices in the partite set to which it does not belong.

Since the four vertices of any partite set cannot be dominated by a single vertex, it follows that the four vertices in each partite set must be dominated by exactly two labeled vertices. In particular, some vertex of G must be labeled 4; in fact, some vertex of degree 2 in G must be labeled 4. We may assume that x_1 is labeled 4. Hence x_1 dominates y_4 only. Necessarily, the remaining three vertices x_1, x_3 and y_2 of U must be dominated by a single vertex. The only possibility for this is to have x_2 labeled 1. Therefore, the vertices of U are dominated by x_1 and x_2. Consequently, each of the four vertices of W must be dominated by a vertex w labeled 2 or a vertex u labeled 3. Necessarily, $w \in W$ and $u \in U$. Since only u can dominate w, it follows that $d(u, w) = 3$. If $w = x_4$ or $w = y_1$, then w does not dominate either x_4 or y_1 and no vertex labeled 3 can dominate both x_4 and y_1. Therefore, $w \neq x_4$ and $w \neq y_1$. Since $x_2 \in W$ has already been labeled 1, it follows that $w = y_3$. Since the only vertex at distance 3 from y_3 in G is x_1 and x_1 is already labeled, it is impossible that $u = x_1$. This implies that $w = y_3$ is not dominated by any labeled vertex, which is a contradiction. □

In order to describe additional grids having an irregular dominating labeling, we first verify the following lemma.

Lemma 2.2. *For each integer $n \geq 8$, there is an irregular dominating labeling of $P_n = (u_1, u_2, \ldots, u_n)$ such that*

(1) *at least one end-vertex of P_n is not labeled and there is at least one element in $[n-1]$ that is not used as a label and*

(2) *for $2 \leq i \leq n-1$, either u_{i-1} is dominated by a vertex u_k where $k \geq i$ or u_{i+1} is dominated by a vertex u_j where $j \leq i$.*

Proof. We proceed by induction. An irregular dominating labeling of P_8 with the desired property is shown in Figure 5 and so the statement is true for $n = 8$.

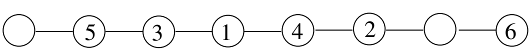

Figure 5: An irregular dominating labeling of P_8

Assume that this statement is true for some path P_n of order $n \geq 8$. We show that this statement is true for the path P_{n+1}. Let $P_{n+1} = (u_1, u_2, \ldots, u_n, u_{n+1})$. Then $P_{n+1} - u_{n+1} = (u_1, u_2, \ldots, u_n) \cong P_n$. By the induction hypothesis, there is an irregular dominating labeling f' of P_n using elements from the set $[n-1]$ such that

(i) at least one end-vertex of P_n is not labeled and there is $t \in [n-1]$ that is not used as a label by f' and

(ii) for $2 \le i \le n-1$, either u_{i-1} is dominated by a vertex u_k where $k \ge i$ or u_{i+1} is dominated by a vertex u_j where $j \le i$.

We may assume that say u_1 is not labeled by f'. We now extend the labeling f' of the path P_n to a labeling f of P_{n+1} by assigning the label n to u_1. We show that f is an irregular dominating labeling of P_{n+1}. Let L' be the set of all vertices of P_n labeled by f'. Then $L = L' \cup \{u_1\}$ is the set of all vertices of P_{n+1} labeled by f. If $v = u_i$ for some integer i with $1 \le i \le n$, then v is dominated by a vertex in L'. If $v = u_{n+1}$, then $d(v, u_1) = n$ and $f(u_1) = n$ and so v is dominated by $u_1 \in L$.

It remains to show that f has the desired properties. First, the end-vertex u_{n+1} of P_{n+1} is not labeled by f. By (i), there is an element $t \in [n-1]$ that is not used by f'. Since $L = L' \cup \{u_1\}$ and $f(u_1) = n$, it follows that t is also not used by f. It remains to show that for $2 \le i \le n$, either u_{i-1} is dominated by a vertex u_k where $k \ge i$ or u_{i+1} is dominated by a vertex u_j where $j \le i$.

- ⋆ If $2 \le i \le n-1$, then it follows by (ii) that either u_{i-1} is dominated by a vertex u_k where $k \ge i$ or u_{i+1} is dominated by a vertex u_j where $j \le i$.
- ⋆ If $i = n$, then u_{n+1} is dominated by u_1.

Consequently, the irregular dominating labeling f of P_{n+1} has the desired properties. $\square$

We are now in a position to describe all those grids $P_n \,\square\, P_2$, $n \ge 3$, having an irregular dominating labeling.

Theorem 2.3. *For each integer $n \ge 3$, the grid $P_n \,\square\, P_2$ has an irregular dominating labeling except when $n = 4$.*

Proof. Since the grid $P_n \,\square\, P_2$ has an irregular dominating labeling for $n = 3, 5, 6, 7$, we may assume that $n \ge 8$. Let $G = P_n \,\square\, P_2$ where $P = (u_1, u_2, \ldots, u_n)$ and $Q = (v_1, v_2, \ldots, v_n)$ are two copies of P_n in G and $u_i v_i \in E(G)$ for $i = 1, 2, \ldots, n$. Let f_0 be an irregular dominating labeling of $P = (u_1, u_2, \ldots, u_n)$ with the properties described in Lemma 2.2 and let U_0 be the set of labeled vertices of P. We may assume that u_1 is not labeled by f_0. Let $t \in [n-1]$ such that t is not used by f_0. We now define a labeling $f : U_0 \cup \{u_1, v_{t+1}\} \to [n]$ of G by

$$\begin{aligned} f(u_i) &= f_0(u_i), \text{ where } u_i \in U_0 \\ f(u_1) &= n \\ f(v_{t+1}) &= t. \end{aligned}$$

Next, we show that every vertex of G is dominated by a vertex labeled by f. Since f_0 is an irregular dominating labeling of P, each vertex of P is dominated by a vertex labeled by f_0 (and so by f). Since (1) $d(v_n, u_1) = n$ and $f(u_1) = n$ and (2) $d(v_1, v_{t+1}) = t$ and $f(v_{t+1}) = t$, it follows that v_n is dominated by u_1 and v_1 is dominated by v_{t+1}. It remains to show for $2 \le i \le n-1$ that the vertex v_i is dominated by a labeled vertex. By Lemma 2.2, either u_{i-1} is dominated by a vertex u_k where $k \ge i$ or u_{i+1} is dominated by a vertex u_j where $j \le i$. First, suppose that u_{i-1} is dominated by a vertex u_k where $k \ge i$. Since $d(u_{i-1}, u_k) = d(v_i, u_k)$, it follows that v_i is also dominated by u_k. Next, suppose that u_{i+1} is dominated by a vertex u_j where $j \le i$. Since $d(u_{i+1}, u_j) = d(v_i, u_j)$, it follows that v_i is also dominated by u_j. Hence, every vertex of G is dominated by a vertex labeled by f. Hence, f is an irregular dominating labeling of G. □

The irregular dominating labeling of $P_n \,\square\, P_2$ described in the proof of Theorem 2.3 gives rise to an irregular dominating labeling of $P_n \,\square\, P_3$.

Theorem 2.4. *For each integer $n \ge 3$, the grid $P_n \,\square\, P_3$ has an irregular dominating labeling.*

Proof. For $3 \le n \le 6$, an irregular dominating labeling of $P_n \,\square\, P_3$ is shown in Figure 6. Thus, we may assume that $n \ge 7$.

Let $G = P_n \,\square\, P_3$ where $Q_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ is a copy of P_n in G for $1 \le i \le 3$ and $u_{i,j}u_{i+1,j} \in E(G)$ for $i = 1, 2$ and $1 \le j \le n$. Then $H = G - V(Q_1) \cong P_n \,\square\, P_2$. Let f_0 be the irregular dominating labeling of H described in the proof of Theorem 2.3 such that (i) $u_{2,1}$ is not labeled and (ii) $f_0(u_{3,t+1}) = t \in [n-1]$. Let L_0 be the set of labeled vertices of H. We now define a labeling $f : L_0 \cup \{u_{2,1}, u_{3,n}\} \to [n+1]$ of G by $f(u) = f_0(u)$ if $u \in L_0$, $f(u_{2,1}) = n$, and $f(u_{3,n}) = n + 1$. Observe that

(1) $f(u_{2,1}) = n$ and so $u_{2,1}$ dominates $u_{1,n}$ and $u_{3,n}$,

(2) $f(u_{3,n}) = n + 1$ and so $u_{3,n}$ dominates $u_{1,1}$, and

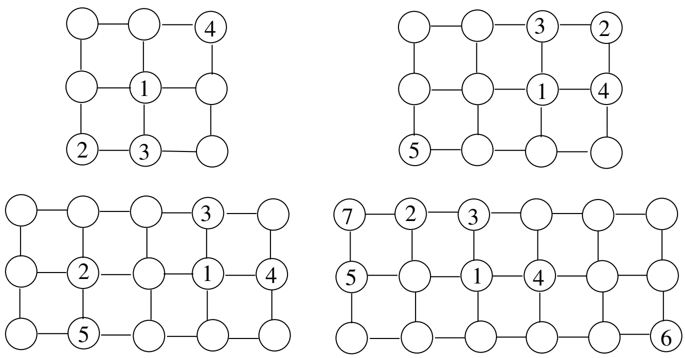

Figure 6: Irregular dominating labelings of $P_n \,\square\, P_3$ for $3 \le n \le 6$

(3) $f(u_{3,t+1}) = t$ and so $u_{3,t+1}$ dominates $u_{3,1}$.

Thus, the corner vertices of G are dominated by labeled vertices of G. Then, by symmetry, an argument similar to the one used in the proof of Theorem 2.3 shows that f is an irregular dominating labeling of G. $\square$

Next, we show for $n \ge 3$ that the grids $P_n \,\square\, P_4$ and $P_n \,\square\, P_5$ have an irregular dominating labeling. It can be shown that every irregular dominating labeling of P_9 must use at least six elements in [8] and so at most two elements in [8] are not used as labels. On the other hand, the irregular dominating labeling of P_{10} shown in Figure 7 uses six labels in [9] and so three elements in [9] are not used as labels. In fact, an argument similar to the proof of Lemma 2.2 provides the following lemma.

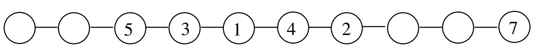

Figure 7: An irregular dominating labeling of P_{10}

Lemma 2.5. *For each integer $n \ge 10$, there is an irregular dominating labeling of $P_n = (u_1, u_2, \ldots, u_n)$ such that*

(1) *at least one end-vertex of P_n is not labeled and there are at least three elements in $[n-1]$ that are not used as labels and*

(2) *for $2 \le i \le n-1$, either u_{i-1} is dominated by a vertex u_k where $k \ge i$ or u_{i+1} is dominated by a vertex u_j where $j \le i$.*

We are now prepared to present the following.

Theorem 2.6. *For each integer $n \geq 3$, the grids $P_n \square P_4$ and $P_n \square P_5$ have an irregular dominating labeling.*

Proof. For $3 \leq n \leq 8$, the result is true by Theorems 2.3 and 2.4. Figure 8 shows an irregular dominating labeling f of $P_9 \square P_5$. Since each label belongs to $[11]$ and no vertices in the first row (and the last row) of $P_9 \square P_5$ are labeled by f, it follows that f gives rise to an irregular dominating labeling of $P_9 \square P_4$ by deleting the first row from $P_9 \square P_5$.

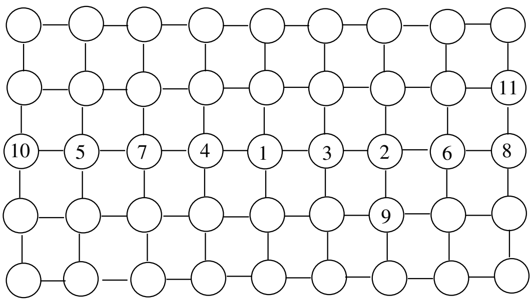

Figure 8: An irregular dominating labeling of $P_9 \square P_5$

Thus, we may assume that $n \geq 10$. We first show that $P_n \square P_5$ has an irregular dominating labeling. Let $G = P_n \square P_5$ where $Q_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ is a copy of P_n in G for $1 \leq i \leq 5$ and $u_{i,j}u_{i+1,j} \in E(G)$ for $1 \leq i \leq 4$ and $1 \leq j \leq n$. Then $\text{diam}(G) = n+3$. We begin with $Q_3 \cong P_n$. By Lemma 2.5, there is an irregular dominating labeling f_0 of Q_3 that satisfies conditions (1) and (2) as described in Lemma 2.5. Let U_0 is the set of labeled vertices of Q_3. We may assume that $u_{3,1}$ is not labeled by f_0. Let $a, b, c \in [n-1]$ that are not used by f_0 such that $a < b < c$. Thus, $a \leq n-3$. We now define a labeling $f : U_0 \cup \{u_{3,1}, u_{2,a+1}, u_{4,b+1}, u_{5,n-c}, u_{2,n-1}, u_{4,1}, u_{5,n}\} \to [n+3]$ of G by

$$\begin{gathered} f(u) = f_0(u), \text{if } u \in U_0,\ f(u_{3,1}) = n, \\ f(u_{2,a+1}) = a, f(u_{4,b+1}) = b,\ f(u_{5,n-c}) = c, \\ f(u_{2,n-1}) = n+1, f(u_{4,1}) = n+2,\ f(u_{5,n}) = n+3. \end{gathered}$$

Next, we show that every vertex of G is dominated by a vertex labeled by f. Since f_0 is an irregular dominating labeling of Q_3, each vertex of Q_3 is dominated by a vertex labeled by f_0 (and so by f). Let

$$U = \{u_{1,1}, u_{1,2}, u_{1,n-1}, u_{1,n}, u_{2,1}, u_{2,n}, u_{4,1}, u_{4,n}, u_{5,1}, u_{5,2}, u_{5,n-1}, u_{5,n}\}.$$

Then every vertex of U is dominated by a vertex labeled by f by observing that

- ⋆ $f(u_{3,1}) = n$ and so $u_{3,1}$ dominates $u_{1,n-1}$, $u_{2,n}$, $u_{4,n}$, and $u_{5,n-1}$,
- ⋆ $f(u_{2,n-1}) = n + 1$ and so $u_{2,n-1}$ dominates $u_{5,1}$,
- ⋆ $f(u_{4,1}) = n + 2$ and so $u_{4,1}$ dominates $u_{1,n}$,
- ⋆ $f(u_{5,n}) = n + 3$ and so $u_{5,n}$ dominates $u_{1,1}$,
- ⋆ $f(u_{2,a+1}) = a$ and so $u_{2,a+1}$ dominates $u_{1,2}$ and $u_{2,1}$,
- ⋆ $f(u_{4,b+1}) = b$ and so $u_{4,b+1}$ dominates $u_{5,2}$ and $u_{4,1}$, and
- ⋆ $f(u_{5,n-c}) = c$ and so $u_{5,n-c}$ dominates $u_{5,n}$.

Finally, it remains to show that every vertex $u \in V(G) - (V(Q_3) \cup U)$ is dominated by a labeled vertex. In fact, we show that every such vertex is dominated by a labeled vertex in Q_3. By symmetry, we may assume that either $u = u_{2,i}$ where $2 \le i \le n - 1$ or $u = u_{1,i}$ where $3 \le i \le n - 2$. We consider these two cases.

Case 1. $u = u_{2,i}$, *where* $2 \le i \le n - 1$.

By Lemma 2.5, either $u_{3,i-1}$ is dominated by a vertex $u_{3,k}$, where $k \ge i$ or $u_{3,i+1}$ is dominated by a vertex $u_{3,j}$, where $j \le i$. First, suppose that $u_{3,i-1}$ is dominated by a vertex $u_{3,k}$, where $k \ge i$. Since $d(u_{3,i-1}, u_{3,k}) = d(u_{2,i}, u_{3,k})$, it follows that $u_{2,i}$ is also dominated by $u_{3,k}$. Next, suppose that $u_{3,i+1}$ is dominated by a vertex $u_{3,j}$, where $j \le i$. Since $d(u_{3,i+1}, u_{3,j}) = d(u_{2,i}, u_{3,j})$, it follows that $u_{2,i}$ is also dominated by $u_{3,j}$.

Case 2. $u = u_{1,i}$, *where* $3 \le i \le n - 2$.

Then $2 \le i - 1 \le n - 3$. By Lemma 2.5, either $u_{3,i-2}$ is dominated by a vertex $u_{3,k}$, where $k \ge i - 1$ or $u_{3,i+2}$ is dominated by a vertex $u_{3,j}$, where $j \le i + 1$. First, suppose that $u_{3,i-2}$ is dominated by a vertex $u_{3,k}$, where $k \ge i - 1$. Since $d(u_{3,i-2}, u_{3,k}) = d(u_{1,i}, u_{3,k})$, it follows that $u_{1,i}$ is also dominated by $u_{3,k}$. Next, suppose that $u_{3,i+2}$ is dominated by a vertex $u_{3,j}$, where $j \le i+1$. Since $d(u_{3,i+2}, u_{3,j}) = d(u_{1,i}, u_{3,j})$, it follows that $u_{1,i}$ is also dominated by $u_{3,j}$.

Therefore, f is an irregular dominating labeling of $G \cong P_n \,\Box\, P_5$.

We now consider $P_n \square P_4$. Let f be the irregular dominating labeling of $G = P_n \square P_5$, as described above. Let U be the set of labeled vertices of G. Observe that (1) no vertices of Q_1 in G are labeled by f and (2) the vertex labeled $n+3$, namely $u_{5,n}$, dominates only $u_{1,1}$. Let

$$F = G - V(Q_1) \cong P_n \square P_4.$$

Then the labeling f gives rise to an irregular dominating labeling

$$g : V(F) \to [n+2]$$

of F defined by $g(u) = f(u)$ for each $u \in U - \{u_{5,n}\}$ with all other vertices unlabeled. $\square$

We are now in a position to determine all subgrids $P_m \square P_n$ of the 8×8 grids that possess an irregular dominating labeling with $2 \le n \le m \le 8$.

Proposition 2.7. *For integers m and n with $2 \le n \le m \le 8$, the $m \times n$ grid $P_m \square P_n$ has an irregular dominating labeling except when $(m, n) \in \{(2, 2), (4, 2)\}$.*

Proof. By Theorems 2.3, 2.4, and 2.6, we may assume that $n \ge 6$. Figure 9 shows an irregular dominating labeling of $P_6 \square P_6$. Thus, we may assume that $m \in \{7, 8\}$ and $n \in \{6, 7\}$.

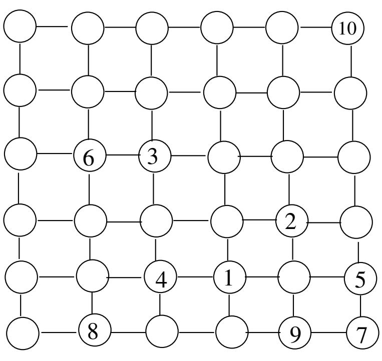

Figure 9: Irregular dominating labelings of $P_6 \square P_6$

First, suppose that $m = 7$. Let $G_n = P_7 \square P_n$, where

$$Q_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,7})$$

is a copy of P_7 in G_n for $1 \le i \le n$ and $u_{i,j}u_{i+1,j} \in E(G)$ for $1 \le i \le n-1$ and $1 \le j \le 7$.

- ⋆ For $n = 6$, an irregular dominating labeling f_6 of G_6 can be defined by $f_6(u_{3,1}) = 6$, $f_6(u_{3,3}) = 2$, $f_6(u_{3,4}) = 4$, $f_6(u_{3,5}) = 1$, $f_6(u_{3,6}) = 3$, $f_6(u_{3,7}) = 5$, $f_6(u_{4,7}) = 7$, $f_6(u_{1,4}) = 8$, and $f_6(u_{5,6}) = 9$ with all other vertices of G_6 not labeled.

- ⋆ For $n = 7$, an irregular dominating labeling f_7 of G_7 can be defined by $f_7(u_{4,1}) = 6$, $f_7(u_{4,3}) = 2$, $f_7(u_{4,4}) = 4$, $f_7(u_{4,5}) = 1$, $f_7(u_{4,6}) = 3$, $f_7(u_{4,7}) = 5$, $f_7(u_{5,7}) = 7$, $f_7(u_{2,4}) = 8$, $f_7(u_{7,4}) = 9$, and $f_7(u_{6,7}) = 10$ with all other vertices of G_7 not labeled.

Next, suppose that $m = 8$. Let $H_n = P_8 \,\square\, P_n$, where

$$T_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,8})$$

is a copy of P_8 in H_n for $1 \leq i \leq n$ and $u_{i,j}u_{i+1,j} \in E(G)$ for $1 \leq i \leq n-1$ and $1 \leq j \leq 8$.

- ⋆ For $n = 6$, an irregular dominating labeling h_6 of H_6 can be defined by $h_6(u_{1,7}) = 11$, $h_6(u_{2,8}) = 8$, $h_6(u_{4,1}) = 9$, $h_6(u_{4,2}) = 5$, $h_6(u_{4,3}) = 3$, $h_6(u_{4,4}) = 1$, $h_6(u_{4,5}) = 4$, $h_6(u_{4,6}) = 2$, $h_6(u_{4,8}) = 6$, $h_6(u_{6,3}) = 10$, and $h_6(u_{6,4}) = 7$ with all other vertices of H_6 not labeled.

- ⋆ For $n = 7$, an irregular dominating labeling h_7 of H_7 can be defined by $h_7(u_{1,8}) = 13$, $h_7(u_{3,8}) = 8$, $h_7(u_{5,1}) = 9$, $h_7(u_{5,2}) = 5$, $h_7(u_{5,3}) = 3$, $h_7(u_{5,4}) = 1$, $h_7(u_{5,5}) = 4$, $h_7(u_{5,6}) = 2$, $h_7(u_{5,7}) = 7$, $h_7(u_{5,8}) = 6$, $h_7(u_{6,1}) = 11$, $h_7(u_{6,8}) = 10$, and $h_7(u_{7,2}) = 12$ with all other vertices of H_7 not labeled. □

Consequently, there is a solution to the $(m+n-2)$ princes problem on the $m \times n$ chessboard for $2 \leq n \leq m \leq 8$ except when $(m, n) \in \{(2, 2), (4, 2)\}$ by Propositions 1.1, 2.1 and 2.7.

3 Concluding remarks

Another well-known chessboard problem deals with the chess piece knight.

> **The knight's tour problem** *Can a knight make a round trip on an 8×8 chessboard visiting each square exactly once?*

Such a round trip is called a *knight's tour*. The answer to this problem is *yes* and both the problem and answer have been known for centuries. A discussion of this problem is given in [12]. Because of the knight's tour problem and its known solution, it follows that for the graph $G = P_8 \square P_8$, there exists a cycle $C = (v_1, v_2, \ldots, v_{64}, v_{65} = v_1)$, visiting every vertex of G exactly once such that every edge v_iv_{i+1} $(1 \leq i \leq 64)$ of C corresponds to a move of a knight on the square v_i to the square v_{i+1}. Necessarily, the distance $d(v_i, v_{i+1}) = 3$ between v_i and v_{i+1} for each $i = 1, 2, \ldots, 64$ on the graph G. Since a 3-prince on a vertex x of a graph can only dominate a vertex y if $d(x, y) = 3$, it follows that there exists a 3-Prince's tour on the 8×8 chessboard as well. Indeed, a tour by a 3-prince is less restrictive since an edge v_iv_{i+1} in a resulting cycle C made by a 3-prince allows v_i and v_{i+1} to be in the same row or column, which is not permitted by a knight in a knight's tour. This, however, brings up the following question.

Problem 3.1. *For which integers k with $1 \leq k \leq 14$, does there exist a k-prince's tour on an 8×8 chessboard?*

If there is a k-prince's tour on an 8×8 chessboard, then for the graph $G = P_8 \square P_8$, there is a cycle $C = (v_1, v_2, \ldots, v_{64}, v_{65} = v_1)$ where $d(v_i, v_{i+1}) = k$ for each $i = 1, 2, \ldots, 64$. We have already seen that such a tour exists if $k = 3$ and clearly such a tour exists for $k = 1$. No such tour exists if k is even since G is a bipartite graph and every two vertices v_i and v_{i+1} for which $d(v_i, v_{i+1}) = k$ must belong to the same partite set of G. Furthermore, since the radius of G is 8, no k-prince's tour is possible for $k \geq 9$. Consequently, we are left with only two questions. *Does there exist a 5-prince's tour and a 7-prince's tour on an 8×8 chessboard?* In the case of a 5-prince's tour, the answer is yes. Such a tour is given in Figure 10. Whether there exists a 7-prince's tour is not known to us.

21	6	29	10	55	36	59	44
14	25	16	31	34	49	40	51
9	18	3	22	43	62	47	56
28	1	20	7	58	45	64	37
5	32	13	26	39	52	33	60
24	15	30	11	54	35	50	41
19	8	17	2	63	48	57	46
12	27	4	23	42	61	38	53

Figure 10: A 5-prince's tour on an 8×8 chessboard

In conclusion, let us return to the fourteen princes problem. The solution given in Figure 2 has the characteristic that if any prince is removed from the chessboard, then not all squares are attacked. This suggests other problems.

Problem 3.2. *Can the* 64 *squares of an* 8×8 *chessboard be attacked without using all* 14 *princes? If so, which prince* (*or princes*) *can be avoided?*

Problem 3.3. *What is the minimum number of distinct princes that can be placed on an* 8×8 *chessboard so that all squares of a chessboard are attacked?*

In addition to the solutions of the $(m + n - 2)$ princes problems given in Theorems 2.3, 2.4, 2.6, and Proposition 2.7, there are solutions to many other $(m+n-2)$ princes problems. This suggests the following conjecture.

Conjecture 3.4. *All* $(m+n-2)$ *princes problems on an* $m\times n$ *chessboard for* $m \geq n \geq 2$ *have a solution except when* $(m, n) \in \{(2, 2), (4, 2)\}$.

References

[1] A. Ali, G. Chartrand, and P. Zhang, *Irregularity in Graphs*, Springer, (2021).

[2] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris* **247** (1958) 258-259.

[3] P. Broe, G. Chartrand, and P. Zhang, Irregular orbital domination in graphs, *Int. J. Comput. Math: Computer Systems Theory*, **7** (2022), 68–79.

[4] G. Chartrand, T.W. Haynes, M.A. Henning, and P. Zhang, *From Domination to Coloring: Stephen Hedetniemi's Graph Theory and Beyond*, Springer, 2019.

[5] G. Chartrand, M.A. Henning, and K. Schultz, On orbital domination numbers of graphs, *J. Combin. Math. Combin. Comput*, **37** (2001), 3–26.

[6] E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi, Total domination in graphs. *Networks* **10(3)** (1977), 211–219.

[7] E.J. Cockayne and S.T. Hedetniemi, Towards a theory of domination in graphs. *Networks* **7** (1977), 247–261.

[8] L. Hayes, K. Schultz, and J. Yates, Universal domination sequences of graphs, *Util. Math*, **54** (1998), 193–209.

[9] T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, *Domination in Graphs: Core Concepts*, Springer, 2023.

[10] M.A. Henning and A. Yeo, *Total Domination in Graphs*, Springer, 2013.

[11] O. Ore, *Theory of Graphs*, *Theory of Graphs*, American Mathematical Society Colloquium Publications, vol. 38, 1962.

[12] J.J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton university press, 2012.

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