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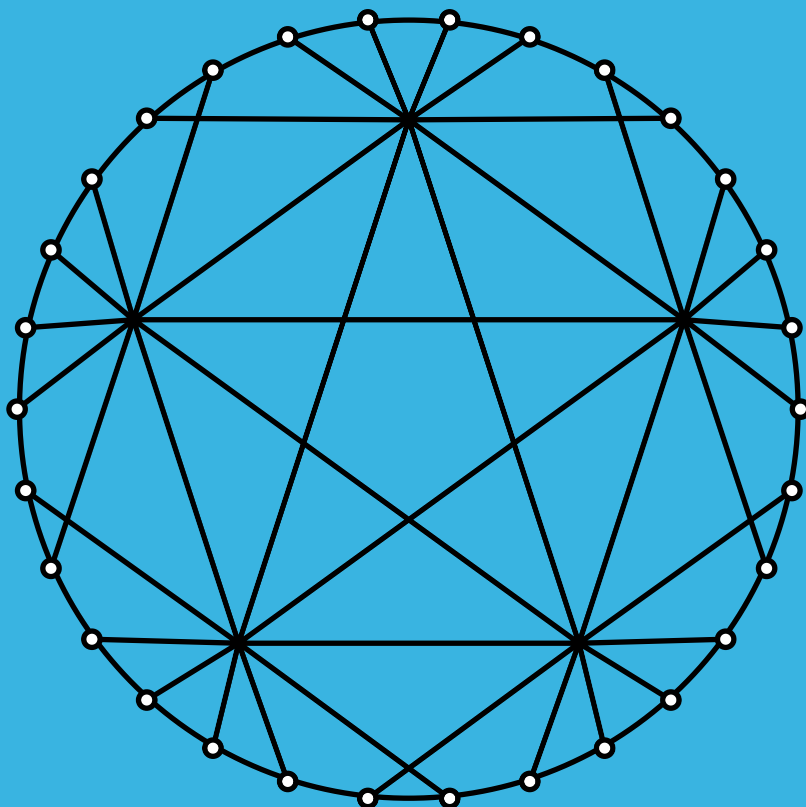

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A tale of two graphs

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Once upon a time – well, actually in 1988, an article by the British author and mathematics educator David Wells [13] appeared in the journal *The Mathematical Intelligencer* that asked the readers to evaluate 24 theorems he listed for their beauty. Two years later, Wells [14] reported the outcome of this survey. The theorem that came out on top was $e^{\pi i}+1=0$, a theorem by the famous Swiss mathematician Leonhard Euler (1707–1783) that gives an amazing relationship among perhaps the five most important numbers 0, 1, π, e, i in all of mathematics. The theorem that came in second was another theorem by Euler, often known as the Euler polyhedron formula (which is actually an identity rather than a formula).

The Euler identity

For any polyhedron with V vertices, E edges, and F faces,

$$V - E + F = 2.$$

The Euler identity is illustrated in Figure 1 for the tetrahedron, cube, and octahedron.

Euler reported this observation to the German mathematician Christian Goldbach (1690–1764) in a letter dated November 14, 1750. (Goldbach is famous for his conjecture that every even integer greater than 2 is the sum of two primes.) This identity only appeared in print (using different notation) two years later (in 1752) in two papers by Euler. In the first of these two papers, Euler stated that he was unable to give a proof of the identity. In the second paper, however, he presented a proof by dissecting polyhedra into tetrahedra. While his proof was clever, it contained flaws.

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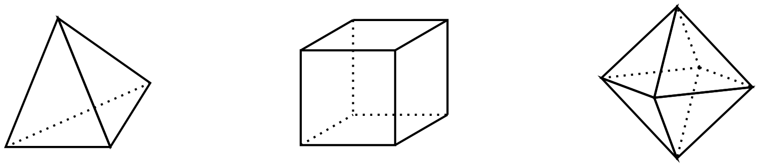


tetrahedron | cube | octahedron

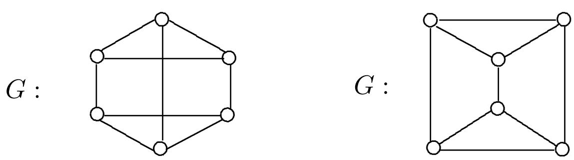
$V = 4,\ E = 6,\ F = 4$ | $V = 8,\ E = 12,\ F = 6$ | $V = 6,\ E = 12,\ F = 8$

$V{-}F{+}F{=}4{-}6{+}4{=}2$ | $V{-}F{+}F{=}8{-}12{+}6{=}2$ | $V{-}F{+}F{=}6{-}12{+}8{=}2$

Figure 1: The tetrahedron, cube and octahedron

The first generally accepted complete proof of this identity was obtained by the famous French mathematician Adrien-Marie Legendre (1752–1833), who is one of 72 engineers, scientists, and mathematicians whose names are engraved on the Eiffel Tower in Paris.

This identity not only holds for polyhedra, it holds for more general structures. It has a prominent place in the area of graph theory. A graph G is *planar* if G can be drawn in the plane without any of its edges crossing. For example, the graph G in Figure 2(a) can be drawn as in Figure 2(b) so that none of its nine edges cross, and so G is planar.



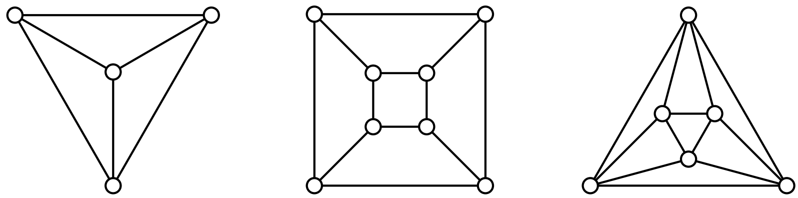
(a) (b)

Figure 2: A planar graph

If G is a connected planar graph (that is, there is a path between every two vertices of G) with n vertices, m edges, and r regions (including the exterior region), then the Euler identity can be restated for connected planar graphs.

Theorem 1 (The Euler identity)**.** *If G is a connected planar graph with n vertices, m edges, and r regions, then $n - m + r = 2$.*

The three polyhedra in Figure 1 can be represented as the planar graphs shown in Figure 3, where one face corresponds to the exterior region of the graph.



graph of the tetrahedron   graph of the cube   graph of the octrahedron

Figure 3: The graphs of the tetrahedron, cube and octahedron

This is the story of two graphs, however, where both graphs entered mathematical history separately and with little fanfare but together turned out to have great relevance in graph theory. These two graphs appeared during attempts to solve two recreational problems, each of which occurred long ago.

The famous German mathematician August Ferdinand Möbius (1790–1868) is probably best known for the Möbius strip, an object that can be constructed from a rectangular piece of paper by twisting one end through 180 degrees and then gluing the two ends together. This object has only one side. While described by Möbius in late 1858, it turns out that it had already been constructed earlier by Johann Benedict Listing (1808–1882), another German mathematician and doctoral student of the famous Carl Friedrich Gauss (1777–1855), who was the first to use the term "topology" in mathematics. Benjamin Gotthold Weiske (1783–1836), a literary scholar and friend of Möbius, mentioned a certain problem to Möbius who stated the problem in a geometry lecture he gave in 1840. Here is the problem.

The five princes problem

Long ago, there was a kingdom ruled by a king who had five sons. It was the king's wish that upon his death, this kingdom should be divided into five regions, one region for each son, in such a way that each region would have a boundary in common with each of the other four regions. Can this be done?

It is actually quite easy to *almost* solve this problem quickly. For example, the kingdom might be divided into the five regions numbered 1, 2, 3, 4, 5, as shown in Figure 4.

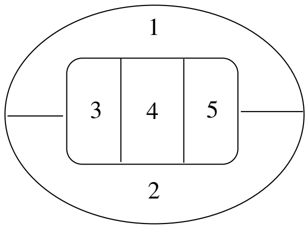


Figure 4: Attempting to solve the five princes problem

In the attempted solution of the Five Princes Problem indicated in Figure 4, every two of the five regions share a common boundary, except regions 3 and 5. So, we are close to a solution but not quite there. In order to give a solution to the problem, we turn to mathematics and to graph theory. Suppose, as the king asked, that the kingdom could be divided into five regions in the desired manner. Then something else would have to be true. Place a point in each region and join two points by a line or curve that passes over the boundary these two regions have in common. This could therefore be done without any two lines or curves crossing each other. Such a point placement is shown in Figure 5 for the attempted solution of the Five Princes Problem given in Figure 4.

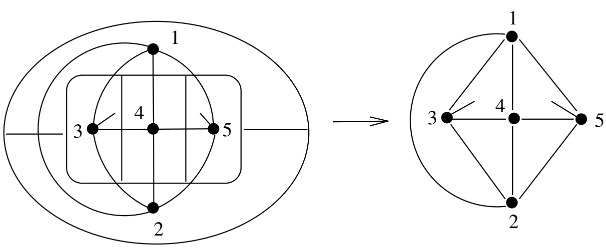


Figure 5: Modeling the kingdom by a graph

If it is possible to divide the kingdom into five regions with this property, then what we have just constructed is a planar graph with five vertices where every two vertices are joined by an edge. If all has gone well, then there are ten edges in all and no two edges cross. With the attempted solution in Figure 5, there appears to be no way to join vertices 3 and 5 by an edge without two edges crossing. The graph of interest is shown in Figure 6. This graph is commonly denoted by K_5 and called the *complete* *graph* of order 5. This graph has order 5 and size 10 (5 vertices and 10 edges). (The graph of the tetrahedron in Figure 3 is the complete graph K_4 of order 4.) If the kingdom in the Five Princes Problem can be divided into five regions as desired, then the graph K_5 would have to be planar. However, if the graph K_5 were planar, it would have to satisfy the Euler identity (Theorem 1). Since K_5 has $n = 5$ vertices, $m = 10$ edges, and r regions, it follows that $n - m + r = 2$ and so $5 - 10 + r = 2$. Therefore, $r = 7$. Let's add the number of edges on the boundary of each region. Since each edge lies on the boundary of two regions, every edge is counted twice and this resulting sum is $2m = 20$. However, there are at least three edges on the boundary of each region and so the sum we obtain is at least $3r = 21$. Therefore, $2m \geq 3r$ and so $20 \geq 21$, which, of course, is impossible. Thus, K_5 is nonplanar and the king's wishes cannot be fulfilled.

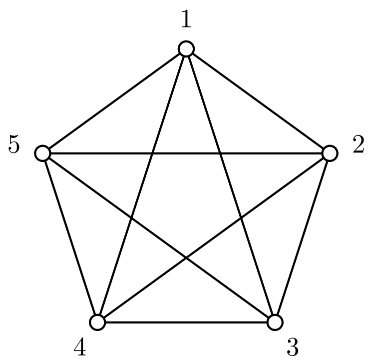


Figure 6: The complete graph K_5 of order 5

We now turn our attention to a second problem – and a second graph.

The three utilities problem

Once there were three houses under construction and each house required connections to each of three utilities, namely, water, electricity and natural gas. Each utility provider requires a direct line from the utility terminal to each house without passing through another provider's terminal or another house along the way. Furthermore, all three utility providers need to bury their lines at the same depth underground without any lines crossing. Can this be done?

While the origin of this problem is unknown, what is known is that this problem dates back more than a century and evidently first appeared in print in a 1913 article written by the British author and puzzle-maker Henry Ernest Dudeney [4] for the *Strand Magazine*. Dudeney belonged to a literary circle in England that included Sir Arthur Conan Doyle, creator of the fictional detective Sherlock Holmes. Four years later (in 1917), Dudeney stated in his book *Amusements in Mathematics* that this problem is as "old as the hills".

The *three utilities problem* is quite clearly a problem in graph theory. The three houses A, B, C and the three utilities w, e, and g (for water, electricity and natural gas, respectively) indicated in Figure 7 is essentially a graph.

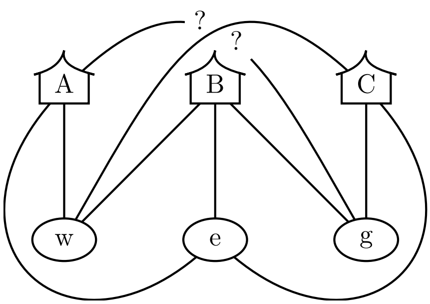


Figure 7: The three utilities problem

This problem is then represented by the graph of Figure 8, denoted by $K_{3,3}$, called the *complete bipartite graph* of order 6 and size 9 whose six vertices are divided into two sets {A, B, C} and {w, e, g} of three vertices each where there is an edge between two vertices if and only if the vertices belong to different sets.

As with the attempted solution of the Five Princes Problem involving the graph K_5, the Three Utilities Problem asks whether the graph $K_{3,3}$ is planar. We nearly have a solution to the Three Utilities Problem with the graph $K_{3,3}$ in Figure 8, where only the utility line (edge) joining the vertices A and g is missing. However, there appears to be no way to add this edge without two edges crossing. Suppose that the graph $K_{3,3}$ is planar. It too would have to satisfy the Euler identity. Since $K_{3,3}$ has $n = 6$ vertices, $m = 9$ edges, and r regions, we must have $n - m + r = 2$ and so $6 - 9 + r = 2$. Therefore, $r = 5$. In this situation, it is impossible to have

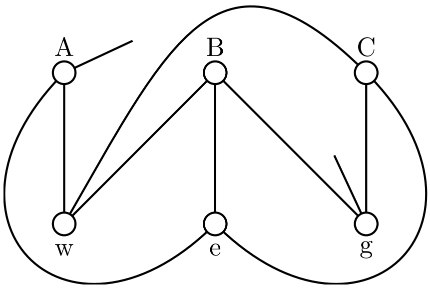


Figure 8: The graph modeling the three utilities problem

a triangular region in any drawing of $K_{3,3}$ in the plane and so there must be at least four edges on the boundary of each region. We now proceed as we did with K_5 and add the number of edges on the boundary of each region over all r regions. Because each edge lies on the boundary of two regions, every edge is counted twice in the sum and the resulting sum is $2m$. On the other hand, since there are at least four edges on the boundary of each of these r regions, the sum we obtain must be at least $4r$ and so $2m \geq 4r$. However, $m = 9$ and $r = 5$; so $2m = 18 \geq 20 = 4r$, which is impossible. Thus, the desired connection between the three houses and the three utilities is impossible, which also means that the graph $K_{3,3}$ is nonplanar.

What we have now seen is that the two graphs K_5 and $K_{3,3}$ are both nonplanar. These graphs are shown in Figure 9 and drawn in their typical manner.

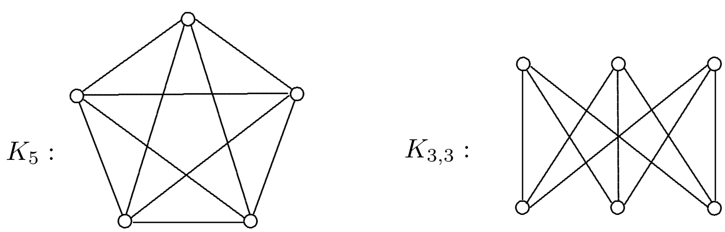


Figure 9: The two nonplanar graphs K_5 and $K_{3,3}$

The graphs K_5 and $K_{3,3}$ are not only nonplanar, it is clear that any graph containing either of these two graphs as a subgraph is also nonplanar. Furthermore, there are graphs similar to K_5 and $K_{3,3}$ that are also nonplanar, such as the graphs in Figure 10. A graph G is a *subdivision* of K_5 or $K_{3,3}$ if G is either K_5 or $K_{3,3}$ or can be obtained from one of these graphs by inserting vertices of degree 2 into one or more edges of the graph. Thus, the graphs in Figure 10 are subdivisions of K_5 and $K_{3,3}$.

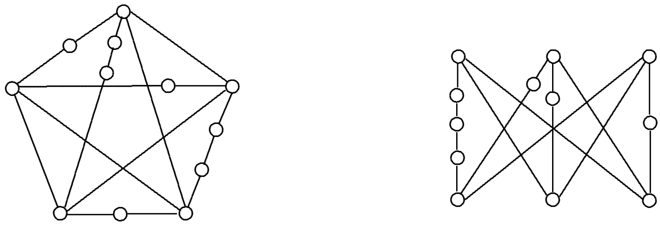


Figure 10: Subdivisions of K_5 and $K_{3,3}$

Quite clearly then, a graph G is not only nonplanar if it contains K_5 or $K_{3,3}$ as a subgraph but it is nonplanar if it were to contain a subdivision of either of these two graphs. This, however, brings up a natural question. If a graph is nonplanar, must it contain a subdivision of K_5 or $K_{3,3}$? It was thought that the answer to this question is *yes* and later it was shown that this is indeed the case. The first published proof of this fact was given in 1930 by the Polish topologist Kazimierz Kuratowski (1896–1980), thereby resulting in the following theorem.

Theorem 2 (Kuratowski's theorem). *A graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

This theorem was first announced in 1929 and the title of Kuratowski's paper is "Sur le problème des courbes gauches en topologie" whose English translation is "On the problem of skew curves in topology". This title suggests, quite correctly, that the setting of Kuratowski's theorem was in topology, not graph theory. Nonplanar graphs were sometimes called *skew graphs* at that time. The publication date of Kuratowski's paper was critical to having the theorem credited to him, for, as it turned out, later in 1930 the two American mathematicians Orrin Frink and Paul Althaus Smith submitted a paper containing a proof of this result as well, but withdrew it after they learned that Kuratowski's paper had preceded theirs, although just barely. Frink and Smith [5] did publish a one-sentence announcement of what they had done in the *Bulletin of the American Mathematical Society* with the title *Irreducible non-planar graphs*. As the title indicates, the setting for their proof was graph theory.

It is believed by some that a proof of this theorem may have been discovered prior to Kuratowski's proof by the Russian topologist Lev Semenovich Pontryagin in his unpublished notes. However, since this possible proof by Pontryagin did not satisfy the established mathematical practice of appearing in print in an accepted refereed journal, the theorem is credited to Kuratowski [10] and to him alone. A more detailed discussion of this story is given in [8].

For a period of time, the only published proof of this famous theorem in graph theory was one expressed in topology rather than graph theory. This changed in the early 1950s, however. The Danish mathematician Gabriel Andrew Dirac (1925–1984) visited the University of Toronto during the academic year 1952–53. In 1952, Dirac had published a paper giving the first sufficient condition for a graph to be Hamiltonian, namely if the degree of every vertex of a graph is at least half of its order, then G has a Hamiltonian cycle. Dirac had a famous stepfather Paul Dirac. He and his contemporary Albert Einstein are considered two of the most prominent physicists of all time. In fact, Paul Dirac was a recipient of the 1933 Nobel Prize in physics. The same year, 1952–53, that Dirac was in Toronto, a graduate student in geometry from Pennsylvania State University went to the University of Toronto as a post-graduate fellow to complete work on his doctoral dissertation. His advisor, the famous geometer H. S. M. Coxeter, was a faculty member there. This student was Seymour 'Sy' Schuster (1926–2020), who had difficulty finding a place to live when he went to Toronto. While there, seeking housing, Schuster met Dirac and after explaining his plight, Dirac offered Schuster the opportunity to stay at his residence until Schuster could find suitable housing. Dirac and Schuster, both in their 20s, discussed mathematics together and the idea occurred of constructing a proof of Kuratowski's theorem of a strictly graph theoretic nature. This is, in fact, what occurred and their published proof [3] appeared in 1954.

Only a few years after the publication of Kuratowski's proof, another event occurred involving the two graphs K_5 and $K_{3,3}$.

If two adjacent vertices u and v in a graph G are identified, then the edge uv is said to be *contracted*. For the graph G of Figure 11, the graph G' is obtained by contracting the edge uv in G while G'' is obtained by contracting the edge wy in G'.

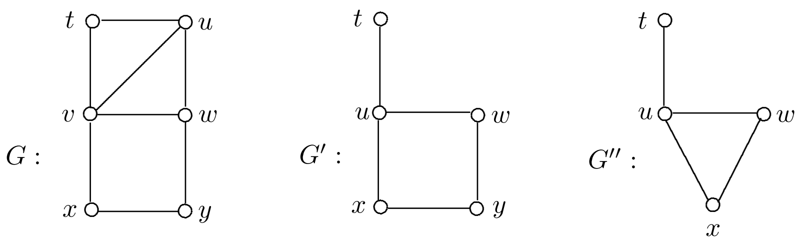


Figure 11: Contracting an edge

A graph H is called a *minor* of a graph G if either H is isomorphic to G or is isomorphic to a graph obtained from G by a succession of edge contractions, edge deletions, and vertex deletions in any order. An immediate observation is the following.

Theorem 3. *If a graph G is a subdivision of a graph H, then H is a minor of G.*

From this theorem, the following corollary results.

Theorem 4. *If G is a nonplanar graph, then K_5 or $K_{3,3}$ is a minor of G.*

The German mathematician Klaus Wagner (1910–2000) received his Ph.D. in 1936 from Universität zu Köln (University of Cologne). That year, 1936, a book was published that is often considered the first textbook on graph theory [9]. A much lesser known book on graph theory [11] was published ten years earlier, however, by the French mathematician André Sainte-Laguë. This 1926 book has been referred to as *the zeroth book on graph theory* by Martin Golumbic [6]. The author of the 1936 book was Dénes König (1884–1944), a Hungarian mathematician who was responsible for many of his students becoming interested in graph theory, including Paul Erdős, one of the best known mathematicians of the 20th century. Mathematician Paul Erdős and physicist Albert Einstein met only once and discussed neither mathematics nor physics, but religion. One year later, in 1937, Wagner [12] was successful in showing that the converse of Theorem 4 was true as well, thereby giving a new characterization of planar graphs.

Theorem 5 (Wagner's theorem)**.** *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Perhaps the best known graph in all of graph theory is the Petersen graph, named for the Danish mathematician Julius Petersen (1839–1910). This cubic graph (every vertex has degree 3) has a variety of interesting properties and has shown to be a counterexample to many conjectures (see [1, 2, 7]). This graph is shown in Figure 12.

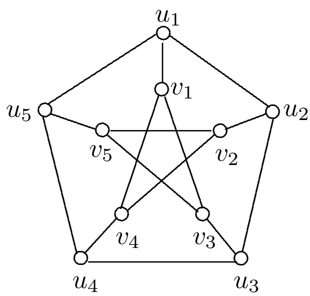


Figure 12: The Petersen graph

One property of the Petersen graph is that it's nonplanar. This graph contains a subgraph that is a subdivision of $K_{3,3}$, as shown in Figure 13. While the Petersen graph does not contain a subdivision of K_5 as a subgraph, the graph K_5 is a minor of the Petersen graph, which is obtained by contracting each of the five edges u_iv_i, $i = 1, 2, \dots, 5$. This once again shows that the Petersen graph is nonplanar. In fact, every cubic graph having K_5 as a minor must contain a subdivision of $K_{3,3}$.

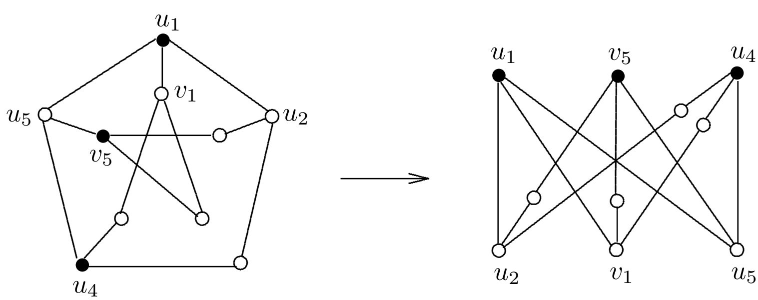


Figure 13: A subdivision of $K_{3,3}$ in the Petersen graph

It has been said that the notation K_5 and $K_{3,3}$ used for these two graphs as well as all complete graphs and all complete bipartite graphs was chosen as these were the initials of Kazimierz Kuratowski. As we look back at what we've just seen, we now have the solution to another problem.

For which nonplanar graphs is the sum of their order and size minimum?

This minimum sum is 15 and there are two solutions to this problem. One solution is K_5. The other is $K_{3,3}$.

Acknowledgement

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