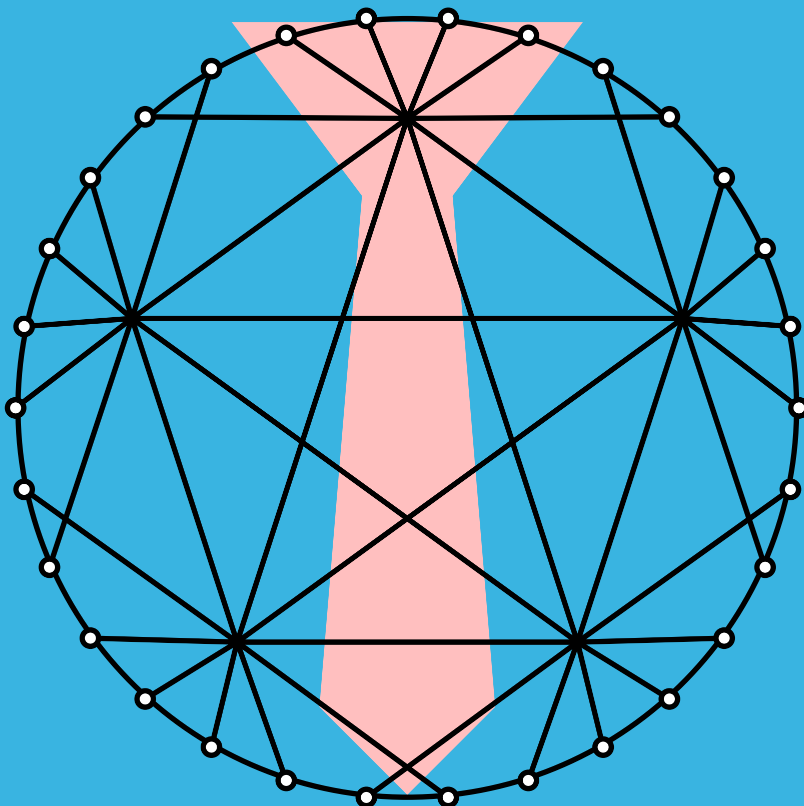
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 99
October 2023



Duluth, Minnesota, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **99** (2023), 133–152

Zonal graphs revisited

Gary Chartrand, Cooroo Egan, and Ping Zhang*

In memory of Ralph Gordon Stanton (1923–2010) on the 100th anniversary of his birth

Abstract. A labeling of the vertices of a connected plane graph G with the two nonzero elements of the ring $\mathbb{Z}_3$ of integers modulo 3 such that the sum of the labels of the vertices on the boundary of every region of G is the zero element of $\mathbb{Z}_3$ is called a zonal labeling and such a graph possessing a zonal labeling is a zonal graph. Several results dealing with zonal graphs are discussed, especially those dealing with proper edge colorings of cubic maps.

1 Introduction

There are many areas within graph theory where the vertices or edges of a graph G (or regions if G is a plane graph) are assigned elements of a set (usually nonnegative integers) in such a way that a desired outcome is produced. Such an assignment is ordinarily referred to as a graph labeling or a graph coloring. Typically, the goal of such an assignment is to minimize the number of elements used to accomplish the goal – but not always.

One of the best known examples of this is a graceful labeling of a graph, a concept introduced by Rosa [11] in 1967, although the terminology is due to Golomb [8] in 1972. For a graph G of order n and size m, a *graceful labeling* f of G is an assignment of distinct integers from the set $\{0, 1, \ldots, m\}$ to distinct vertices of G so that an edge uv of G is assigned the label

*Corresponding author: ping.zhang@wmich.edu
Key words and phrases: plane graph, zonal labeling, zonal graph, cubic maps, proper edge colorings, the Four Color Theorem.
AMS (MOS) Subject Classifications: 05C10, 05C15, 05C38, 05C78.

Received: 1 July 2022
Accepted: 29 October 2022

$|f(u) - f(v)|$ in such a way that distinct edges have distinct labels. A graph possessing a graceful labeling is called a *graceful graph*. Many results dealing with graceful and related labelings have been obtained and many of these results have been described in a periodically updated survey due to Gallian [9]. A popular conjecture dealing with graceful graphs is the following due to Kotzig and Ringel [9].

The graceful tree conjecture *Every nontrivial tree is graceful.*

In 1980, Graham and Sloane [10] introduced another graph labeling called a harmonious labeling. For a connected graph G of order n and size m, a *harmonious labeling* f of G is an assignment of distinct elements from the ring $\mathbb{Z}_m$ of integers modulo m to the vertices of G so that an edge uv of G is assigned the label $f(u) + f(v)$ in $\mathbb{Z}_m$ in such a way that distinct edges have distinct labels in $\mathbb{Z}_m$. For $m = n - 1$, the graph G is a tree and such a labeling is impossible. In this case, some element of $\mathbb{Z}_m$ is assigned to two vertices of G, while all other elements of $\mathbb{Z}_m$ are used exactly once. A connected graph possessing a harmonious labeling is a *harmonious* graph. From this concept, the following corresponding conjecture arose.

The harmonious tree conjecture *Every nontrivial tree is harmonious.*

In 1986, another labeling concept was introduced, where positive integers were assigned to the edges of a graph in such a way that the resulting vertex labels obtained by adding the labels of the incident edges are distinct. The *irregularity strength* of a graph G is the smallest positive integer k such that if each edge of G is assigned one of the labels from the set $[k] = \{1, 2, \dots, k\}$, then distinct vertices have distinct labels (see [4]). Interpreting vertex labels in this way as colors led to a corresponding proper coloring when in 2004 Karoński, Łuczak, and Thomason [6] stated the following conjecture.

The 1-2-3 conjecture *For every connected graph G of order 3 or more, each edge of G can be assigned one of the labels $1, 2, 3$ in such a way that the vertex colors obtained by adding the labels of the incident edges of every two adjacent vertices are different.*

While it was shown in [7] that there is a corresponding 1-2-3-4-5 theorem, no 1-2-3-4 theorem is known at present.

Probably the most common and oldest examples of such labelings or colorings are proper colorings of a graph where the vertices are assigned positive integer colors from a set $[k]$ for some positive integer k in such a way that every two adjacent vertices are assigned distinct colors. The goal here is to determine the minimum positive integer k for which this is possible, resulting in the *chromatic number* of the graph. Certainly the best known problem dealing with this topic concerns planar graphs and is due to Francis Guthrie in 1852, which was ultimately solved by Appel and Haken [1] in 1976. The book by Wilson [15] provides much information on the history and solution of this problem. One way to state this famous resulting theorem in terms of graphs is the following.

The Four Color Theorem *The chromatic number of every planar graph is at most* 4.

The problem (referred to as the *Four Color Problem*), as originally stated by Guthrie, did not deal with graphs at all but with maps, that is, plane graphs, asking whether the regions of every map could be colored with at most four colors in such a way that every two regions with a common boundary line are colored differently.

In 2014, Egan introduced another vertex labeling concept dealing with plane graphs, which was discussed in [3]. This concept involved the use of only two labels.

2 Zonal labelings

Let G be a connected planar graph (or multigraph) embedded in the plane, that is, G is a plane graph. A labeling of the vertices of G with the two nonzero elements 1 and 2 of the ring $\mathbb{Z}_3$ of integers modulo 3 is called a *zonal labeling* of G if the sum of the labels of the vertices on the boundary of every region (or zone) of G, called the *value* of the region, is 0, the zero element in $\mathbb{Z}_3$. If a plane graph G possesses such a labeling, then G is called a *zonal graph*. A planar graph G is said to be *zonal* if there exists a zonal planar embedding of G. As is the case with graceful labelings, harmonious labelings, and many other labelings, the primary question here as well is the following: *Which plane graphs are zonal?*

Before presenting some results dealing with zonal labelings, we state a number of observations. Let there be given a vertex labeling f of a graph G with the labels 1 and 2 of $\mathbb{Z}_3$. The *complementary labeling* $\overline{f}$ of f is defined as $\overline{f}(v) = 3 - f(v)$ for each vertex v of G. We then have the following observation.

Observation 2.1. *If a labeling of the vertices of a plane graph is zonal, then its complementary labeling is also zonal.*

For example, consider the cycles C_3, C_4, C_5, and C_6. All four of these cycles are zonal. As a consequence of Observation 2.1, there is essentially only one way to give a zonal labeling of C_3, essentially only two ways to give a zonal labeling of C_4, essentially only one way to give a zonal labeling of C_5, and essentially only four ways to give a zonal labelings of C_6. These zonal labelings are shown in Figure 1.

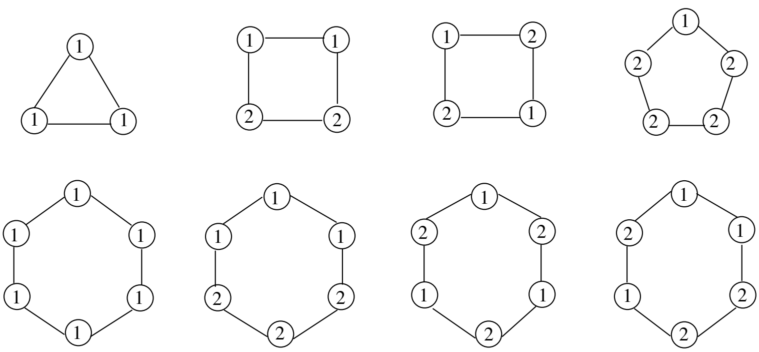


Figure 1: Zonal labelings of C_3, C_4, C_5, and C_6

In fact, every cycle C_n, $n \geq 2$, and every nontrivial tree is zonal (see [3]).

Proposition 2.2. *Every nontrivial tree and every cycle is zonal.*

The zonal labelings of the graphs of Figure 1 illustrate the following observation.

Observation 2.3. *Let H be the boundary of a region R in a plane graph G with a zonal labeling ℓ. If n_i vertices of H are labeled i by ℓ for $i = 1, 2$, then $n_1 \equiv n_2 \pmod 3$.*

Proof. Since the value of a region R in G is 0 in $\mathbb{Z}_3$, it follows that $1 \cdot n_1 + 2 \cdot n_2 \equiv 0 \pmod 3$. Therefore, $n_1 \equiv n_2 \pmod 3$. □

Consider the plane graphs G_1 and G_2 of Figure 2. A zonal labeling of G_1 is given in that figure; therefore, G_1 is zonal. Now consider G_2. If there exists a zonal labeling of G_2, then u_2, v_2, w_2 must all be assigned 1 or all assigned 2. The same is true for x_2, y_2, z_2. Regardless of how this is done, however, the sum of the labels of the vertices on the boundary cycle (a 5-cycle) of the region R in G_2 or on the exterior boundary cycle is not 0 in $\mathbb{Z}_3$. Thus, G_2 is not zonal. Since the graphs G_1 and G_2 are isomorphic, this shows that it's possible for a planar graph to be embedded in the plane in two different ways and obtain two different outcomes. A planar graph G is *absolutely zonal* if every planar embedding of G is zonal and a a zonal planar graph G is *conditionally zonal* if there is a planar embedding of G that is not zonal. These concepts were studied in [2]. Thus, the planar graph in Figure 2 is conditionally zonal. Since the boundary of a planar embedding of a nontrivial tree and a cycle is the graph itself, it follows by Proposition 2.2 that every nontrivial tree and every cycle is absolutely zonal. Whitney [14] showed that every 3-connected planar graph is uniquely embeddable in the plane. Therefore, no 3-connected planar graph is conditionally zonal.

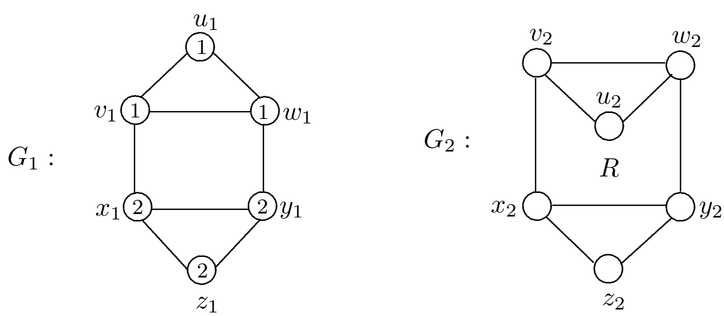


Figure 2: Two plane graphs

3 Zonal labelings and cycles

That there is a connection that zonal labelings of cycles has with proper edge colorings of cycles using three colors is mentioned in [3]. Let $C = (v_1, v_2, \ldots, v_n, v_1)$ be an n-cycle embedded in the plane where the vertices of C are labeled as shown in Figure 3. That is, we assume that we proceed about the interior of C in a counter-clockwise direction.

Thus, after an edge $v_{i-1}v_i$ is encountered, the next edge encountered is v_iv_{i+1}, as indicated in Figure 4. Let there be given a proper coloring c of edges of C with the three colors $1, 2, 3$. If the color $c(v_iv_{i+1})$ of the edge v_iv_{i+1} immediately follows the color $c(v_{i-1}v_i)$ of the edge $v_{i-1}v_i$ numerically

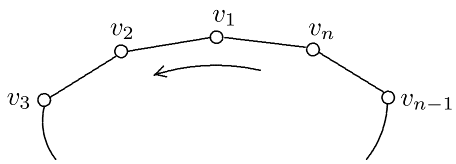


Figure 3: A cycle C

(that is, if 2 follows 1, 3 follows 2, or 1 follows 3), then the vertex v_i on C is said to be of *type* 1. Otherwise, v_i is of *type* 2. See Figure 4 in the case where $c(v_{i-1}v_i) = 1$.

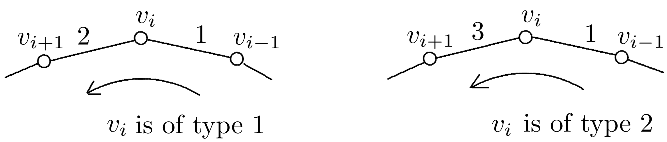


Figure 4: The type of a vertex

Theorem 3.1. *For every proper edge coloring of a cycle C with the colors $1, 2, 3$, the resulting types of the vertices of C produce a zonal labeling of C.*

Proof. There is essentially only one way to properly color the edges of C_3 and essentially only two ways to properly color the edges of C_4 with the colors $1, 2, 3$. These are shown in Figure 5 and the resulting vertex types result in a zonal labeling in each case.

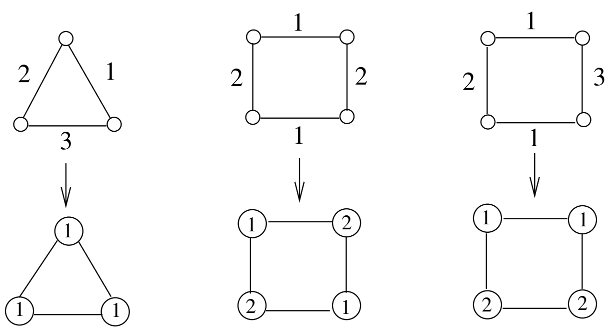


Figure 5: Proper edge colorings of C_3 and C_4 and resulting vertex types

Suppose that the statement is false. Then there is a least integer $n \geq 5$ for which a proper edge coloring of C_n results in vertex types that is not a zonal labeling of C_n. If only two of the three colors $1, 2, 3$ are used in the edge coloring of C_n, then the edge colors alternate in C_n. This implies that n is even and the vertex types alternate in C_n, which results in a zonal labeling of C_n. This produces a contradiction. Consequently, all three colors must be used in the edge coloring of C_n, which implies that C_n has a path of length 3 whose three edges have distinct colors. We may assume that C_n contains the path (u, v_1, v_2, w) where the edge uv_1 is colored 1, the edge v_2w is colored 2, and the edge v_1v_2 is colored 3. We now identify the vertices v_1 and v_2 in C_n, denoting the resulting vertex by v and producing the cycle C_{n-1}. In C_n, the type of the vertices v_1 and v_2 are both 2, while in C_{n-1}, the type of the vertex v is 1. Since the vertex types in C_{n-1} is a zonal labeling of C_{n-1}, the sum of the labels of the vertices of C_{n-1} is 0 in $\mathbb{Z}_3$. Consequently, the sum of the vertex types in C_n is also 0 in $\mathbb{Z}_3$, which implies that the vertex types in C_n is a zonal labeling of C_n. This is a contradiction. $\square$

In fact, the converse of Theorem 3.1 is true as well.

Theorem 3.2. *For each zonal labeling ℓ of a cycle C, there is a proper edge coloring of C with the colors $1, 2, 3$ such that the resulting type of each vertex v of C is $\ell(v)$.*

Proof. For a given zonal labeling ℓ of a cycle C, we define a proper edge coloring of C with the colors $1, 2, 3$ such that the type of each vertex v of C is $\ell(v)$. First, suppose that each vertex of C has the same label. By Observation 2.1, we may assume that every vertex is labeled 1 by ℓ. Then $3 \mid n$ and the coloring $1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3$ of the edges of C in a counter-clockwise direction results in each vertex of C having type 1.

Next, suppose that both labels 1 and 2 are used in the zonal labeling ℓ of C. Then $n \geq 4$. In this case, we proceed by the Strong Form of Induction on n to define a proper edge coloring of C with the colors $1, 2, 3$ such that the type of each vertex v of C is $\ell(v)$. Since there are adjacent vertices of C with distinct labels, we may assume that $\ell(v_3) = 1$ and $\ell(v_4) = 2$. Let C' be the $(n-2)$-cycle obtained from C by deleting v_3 and v_4 and joining v_2 to v_5. Then $C' = (v_1, v_2, v_5, v_6, \ldots, v_n, v_1)$ is an $(n-2)$-cycle. Let ℓ' be the labeling of C' where $\ell'(v) = \ell(v)$ for each vertex v of C'. Since ℓ is a zonal labeling of C, so is ℓ'. By the induction hypothesis, there is a proper edge coloring of C' with the colors $1, 2, 3$ such that the type of each vertex

v of C' is $\ell'(v)$. We may assume that $c'(v_5v_6) = 1$. We now consider four situations, depending on the labels $\ell'(v_2)$ and $\ell'(v_5)$ in C'.

- ⋆ If $\ell'(v_2) = \ell'(v_5) = 1$, then $c'(v_2v_5) = 3$ and $c'(v_1v_2) = 2$. Define $c(v_4v_5) = 3$, $c(v_3v_4) = 1$ and $c(v_2v_3) = 3$.
- ⋆ If $\ell'(v_2) = \ell'(v_5) = 2$, then $c'(v_2v_5) = 2$ and $c'(v_1v_2) = 3$. Define $c(v_4v_5) = 2$, $c(v_3v_4) = 3$ and $c(v_2v_3) = 2$.
- ⋆ If $\ell'(v_2) = 1$ and $\ell'(v_5) = 2$, then $c'(v_2v_5) = 2$ and $c'(v_1v_2) = 1$. Define $c(v_4v_5) = 3$, $c(v_3v_4) = 1$ and $c(v_2v_3) = 3$.
- ⋆ If $\ell'(v_2) = 2$ and $\ell'(v_5) = 1$, then $c'(v_2v_5) = 3$ and $c'(v_1v_2) = 1$. Define $c(v_4v_5) = 3$, $c(v_3v_4) = 1$ and $c(v_2v_3) = 3$.

In each case, a proper edge coloring of C with the colors $1, 2, 3$ is produced such that the type of each vertex v of C is $\ell(v)$. □

4 Zonal labelings and cubic maps

A natural question concerning zonal labelings asks whether there's any special motivation for studying zonal labelings in graph theory. To give one possible answer to this question, we now turn to the class of plane graphs often called cubic maps. A *cubic map* is a connected bridgeless cubic plane graph (or multigraph). The following result was obtained in [3].

Theorem 4.1. *A connected cubic plane graph G is zonal if and only if G is bridgeless.*

By Theorem 4.1, all cubic maps are absolutely zonal. Clearly, every cubic map has even order. The cubic maps of order 6 or less are shown in Figure 6. A zonal labeling of each such graph is also shown in Figure 6.

Not only does every cubic map have a zonal labeling, its complementary labeling is also a zonal labeling. That is, if the label of every vertex in a cubic map with a zonal labeling is replaced by its complementary label, then another zonal labeling is produced. However, we do not consider these two zonal labelings distinct. Rather than give a universal complementary labeling of a cubic map, it may be possible to give a more localized complementary labeling and produce another zonal labeling. That is, it may

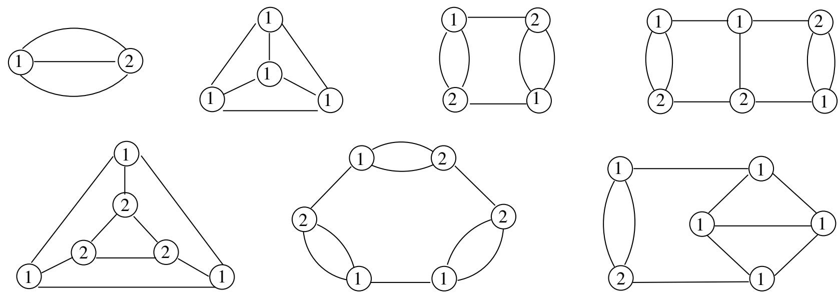


Figure 6: Zonal labelings of cubic maps with two, four, or six vertices

be possible to find a proper subset S of the vertex set of a cubic map M and replace only the label of every vertex of S by its complementary label and produce another zonal labeling of M. For example, consider the zonal labeling of the 3-cube Q_3 shown in Figure 7(a). If the labels of the vertices in the set $\{u, v, w, x\}$ are replaced by their complementary labels, then the resulting labeling of Q_3, shown in Figure 7(b), is also a zonal labeling of Q_3.

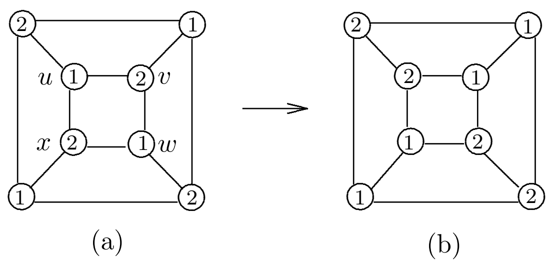


Figure 7: Producing a new zonal labeling of the 3-cube Q_3

The following result shows that in some cases, there are many possibilities for the number of vertices whose labels may be reversed in a cubic map with a zonal labeling to obtain a new zonal labeling.

Theorem 4.2. *For every two even integers r and n such that $n \geq 10$ and $2 \leq r \leq n-2$, there exists a cubic map M of order n with a zonal labeling and an r-element subset S_r of $V(G)$ such that if the labels of all vertices of S_r are replaced by their complementary labels, then the resulting labeling is zonal.*

Proof. Let $n \geq 10$ be an even integer. Then $n = 2k + 8$ for some positive integer k. We construct a cubic map M of order n as follows. Let F

and H be two disjoint copies of the graph $K_4 - e$ such that $V(F) = X = \{x_1, x_2, x_3, x_4\}$ where $e = x_1x_2$ and $V(H) = Y = \{y_1, y_2, y_3, y_4\}$ where $e = y_1y_2$. The vertices x_1 and y_1 are connected in M by the path (x_1, u_1, u_2, …, u_k, y_1) and the vertices x_2 and y_2 are connected in G by the path (x_2, v_1, v_2, …, v_k, y_2). The construction of M is completed by adding the edges u_iv_i for $1 \leq i \leq k$. The cubic map M is embedded in the plane as shown in Figure 8. The cubic map M has order $n = 2k + 8$ where the boundary of the exterior region is an $(n - 2)$-cycle. The boundaries of two interior regions are 5-cycles, the boundaries of four interior regions are 3-cycles, while the boundary of each of the remaining $k - 1$ regions is a 4-cycle.

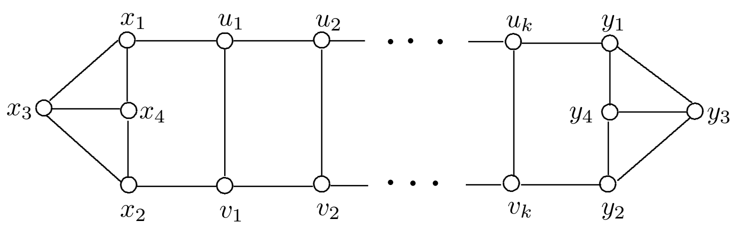


Figure 8: The cubic map G in the proof of Theorem 4.2

Let $V_i = \{u_i, v_i\}$ for $1 \leq i \leq k$. Then $\mathcal{P} = \{X, Y, V_1, V_2, \ldots, V_k\}$ is a partition of $V(M)$. For every even integer r with $2 \leq r \leq n-2$, there exists a subset of $\mathcal{P}$ such that the union S_r of the elements in this subset has $|S_r| = r$. By replacing the labels of all vertices of S_r by their complementary labels, a zonal labeling of M is obtained. $\square$

Certainly, for a cubic map M with a zonal labeling, it is impossible to replace the label of just one vertex v of M with its complementary label to produce a zonal labeling, as doing this give a nonzero value to each of the three regions having v on their boundaries. It is not only impossible to replace the label of exactly one vertex of M by its complementary label to produce a zonal labeling, it is also impossible to replace the labels of exactly three vertices of M by their complementary labels to produce a zonal labeling. To see this, we first present the following lemma.

Lemma 4.3. *No set of three vertices in a cubic map M lie on three different boundary cycles of M.*

Proof. Assume, to the contrary, that there is a cubic map M with three vertices u, v, w, all of which lie on three distinct boundary cycles in M. Let

M be embedded in the plane so that one of these three boundary cycles, say C, is the exterior cycle of M. Let $C = (u = x_1, x_2, \ldots, x_a = v, \ldots, x_b = w, \ldots, x_k, u)$. Let e be the edge incident with u that does not lie on C, let C' be the boundary cycle containing the edges e and x_1x_2, and let C'' be the boundary cycle containing the edges e and x_1x_k. Next, let P' be the path on C' with initial vertex u and initial edge e such that the terminal vertex of P', say y, is the first vertex of P' belonging to C. Now, let Q' be the $u - y$ path of C containing x_1x_2 and let Q'' be the $u - y$ path of C containing x_1x_k. Thus, P' and Q' form a cycle S' and P' and Q'' form a cycle S''. Let M' be the plane subgraph of M induced by the edges lying on and inside S' and let M'' be the plane subgraph of M induced by the edges lying on and inside S''. Hence, C' lies within M' and C'' lies within M''. The subgraphs M' and M'' have only P' in common and u and y are the only vertices of C belonging to both M' and M''. Thus, at most one of C' and C'' contains all three vertices u, v, w, which is a contradiction. $\square$

Proposition 4.4. *For any zonal labeling of a cubic map M, the labeling obtained by replacing the labels of exactly three vertices of M by their complementary labels is not a zonal labeling of M.*

Proof. Assume, to the contrary, that there is a zonal labeling ℓ of a cubic map M for which the labeling ℓ' obtained by replacing the labels of three vertices u, v, w of M by their complementary labels is a zonal labeling of M. Since there is no boundary cycle of a region R in M that contains exactly one of u, v, w, it follows that any boundary cycle in M containing any of the vertices u, v, w must contain exactly two or exactly three of these vertices.

Suppose that there is a boundary cycle C in M containing all three of the vertices u, v, w. Necessarily, these three vertices must have the same label. Since every vertex lies on three boundary cycles, it follows by Lemma 4.3 that there must be a boundary cycle C' of a region R' containing exactly two of u, v, w. However, when the labels of these two vertices on C' are replaced by their complementary labels, the value of R' is nonzero and so ℓ' is not a zonal labeling, a contradiction. Consequently, every boundary cycle of M containing any of the vertices u, v, w contains exactly two of these vertices. This implies that these three vertices cannot have the same label. Thus, we may assume that exactly two of u, v, w have the same label, say $\ell(u) = \ell(v) = 1$ and $\ell(w) = 2$. Hence, no boundary cycle in M contains both u and v. Since u lies on three distinct boundary cycles, say C_u, C'_u, C''_u, of M, it follows that w must lie on these three boundary cycles as well. Since v also lies on three distinct boundary cycles, say C_v, C'_v, C''_v, of M and none of these boundary cycles contains u, it follows that w must lie on these three boundary cycles. However then, w lies on six distinct boundary cycles of M, which is impossible. □

We are led to the following conjecture.

Conjecture 4.5. *For any zonal labeling of a cubic map M and any odd positive integer r, the labeling obtained by replacing the labels of exactly r vertices of M by their complementary labels is not a zonal labeling of M.*

5 Edge colorings of cubic maps

Suppose that G is a cubic map possessing a proper coloring of the edges with the colors 1, 2, 3. With each such edge coloring, one can also define a *type* of each vertex of G, as we did for cycles. Let v be a vertex of G. There are three edges of G incident with v and these three edges are colored 1, 2, 3 in some order as one proceeds clockwise about v. If the colors of the edges are encountered in the order $1-2-3$, then v is said to be *type* 1. If the colors of the edges are encountered in the order $1-3-2$, then v is said to be *type* 2. See Figure 9. Observe that if C is the boundary cycle of a region in G and v is a vertex of C, then the type of the vertex v on the cycle C is the same as the type of v in the cubic map G.

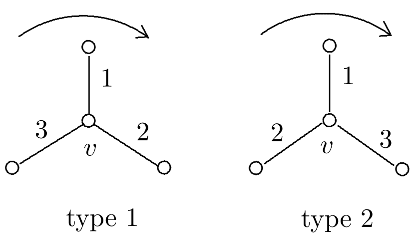


Figure 9: The vertex types in a proper 3-coloring of the edges of a cubic map

For example, the 3-cube $Q_3 = C_4 \square K_2$ as drawn in Figure 10 is a cubic map. A proper edge coloring of Q_3 with the colors 1, 2, 3 is shown in Figure 10 and the vertices of Q_3 are labeled with their types from this edge coloring. This labeling is a zonal labeling of Q_3. Not only do the vertex types of the proper edge coloring of the cubic map Q_3 in Figure 10 produce a zonal labeling of Q_3, this is the case for every cubic map.

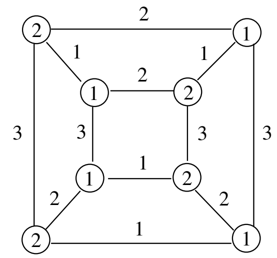


Figure 10: A proper edge coloring and a zonal labeling of Q_3

Theorem 5.1. *The vertex types of every proper edge coloring of a cubic map M with the colors $1, 2, 3$ produce a zonal labeling of M.*

Proof. Let M be a cubic map with a proper edge coloring using the colors $1, 2, 3$. Let C be the boundary cycle of a region of M. We saw that vertex types of the vertices of C are the vertex types of C obtained from the proper edge coloring C using the colors $1, 2, 3$. By Theorem 3.1, this is a zonal labeling of C and so the value of C is 0 in $\mathbb{Z}_3$. Since this is the case for every boundary cycle of a region of M, the vertex types of M produce a zonal labeling of M. $\square$

The following observation will be useful to us.

Lemma 5.2. *Let G be a cubic map with a zonal labeling ℓ. For a cycle C of G, let*

$$S = \{v \in V(C) : G \textit{ contains an edge incident with } v \textit{ lying outside of } C\}$$
$$T = \{v \in V(C) : G \textit{ contains an edge incident with } v \textit{ lying inside of } C\}$$

Then $\displaystyle\sum_{v \in S} \ell(v) + \sum_{v \in T} 2\ell(v) \equiv 0 \pmod 3$.

Proof. For each region R lying within C, let $B(R)$ denote the boundary cycle of R. Since ℓ is a zonal labeling of G, it follows that $\sum_{v \in V(B(R))} \ell(v) \equiv 0 \pmod 3$. Let W be the set of all vertices of G lying interior to C. If $v \in W$, then v lies on exactly three boundary cycles within C; if $v \in S$, then v lies on exactly one boundary cycle within C; and if $v \in T$, then v lies on exactly two boundary cycles within C. Let $\mathcal{R}$ be the set of all regions lying interior to C. Since $\sum_{v\in V(B(R))}\ell(v) \equiv 0 \pmod 3$ for each $R \in \mathcal{R}$, it follows that

$$\sum_{R\in\mathcal{R}}\left(\sum_{v\in V(B(R))}\ell(v)\right) = \sum_{v\in S}\ell(v) + \sum_{v\in T}2\ell(v) + \sum_{v\in W}3\ell(v) \equiv 0 \pmod 3$$

and so $\sum\limits_{v\in S}\ell(v) + \sum\limits_{v\in T}2\ell(v) \equiv 0 \pmod 3$. □

We now show that for a given cubic map G with a zonal labeling ℓ, if some color in $\mathbb{Z}_3 = \{1,2,3\}$ is assigned to an edge e of G, then there is a unique color in $\mathbb{Z}_3$ assigned to each edge in a proper edge coloring of G with elements of $\mathbb{Z}_3$ according to the vertex type $\ell(x)$ of each vertex x of G. Let f be an arbitrary edge of G distinct from e and let $P = (v_1, v_2, \ldots, v_k)$, $k \geq 3$, be a $v_1 - v_k$ path in G, where $e = v_1v_2$ and $f = v_{k-1}v_k$. The vertex type $\ell(v_i)$, $1 \leq i \leq k$, of each vertex v_i of P produces a color from $\mathbb{Z}_3$ for each of the edges $v_2v_3, v_3v_4, \ldots, v_{k-1}v_k$ recursively as we proceed along the path P. This results in a color $c(f)$ for the edge f. It remains to show that if the color of f had been determined in this manner from any other path Q from e to f, we would have exactly the same color $c(f)$ for f. We verify this next. In the first result, we consider the case where Q belongs to a cycle in G. In the second result, we consider the more general situation where Q does not lie on a cycle of G.

Theorem 5.3. *Let G be a cubic map with a zonal labeling ℓ and let C be a cycle of G. If an edge of C is assigned a color from $\{1,2,3\}$ in $\mathbb{Z}_3$, then there is a unique proper edge coloring of C with the colors $1,2,3$ such that the type of each vertex x of C is $\ell(x)$.*

Proof. Let $C = (v_1, v_2, \ldots, v_k, v_{k+1} = v_1)$ be a k-cycle of G, where $k \geq 2$, and let $e \in E(C)$, where the subscript of a vertex of C is expressed as a positive integer modulo k. We may assume that $e = v_kv_1$ and that v_kv_1 is assigned color $c(v_kv_1) \in \{1,2,3\}$. For $1 \leq i \leq k$, let w_i be the neighbor of v_i that is not on C. Furthermore, for $1 \leq j \leq k$, let

$$\begin{aligned} S_j &= \{v_i \in V(C) : 1 \leq i \leq j \text{ and } w_i \text{ lies outside of } C\} \\ T_j &= \{v_i \in V(C) : 1 \leq i \leq j \text{ and } w_i \text{ lies inside of } C\}. \end{aligned}$$

First, we proceed from v_kv_1 to v_jv_{j+1} along the path $(v_k, v_1, v_2, \ldots, v_{j+1})$. According to the label $\ell(v_1)$ of v_1 and the color $c(v_kv_1)$ of v_kv_1, we assign the two colors $\{1,2,3\} - \{c(v_kv_1)\}$ to the two edges v_1v_2 and v_1w_1 incident with v_1 such that the type of v_1 is $\ell(v_1)$. Thus, if v_1w_1 lies exterior to C, then $c(v_1v_2) = c(v_kv_1) + \ell(v_1)$ in $\mathbb{Z}_3$; while if v_1w_1 lies interior to C, then $c(v_1v_2) = c(v_kv_1) + 2\ell(v_1)$ in $\mathbb{Z}_3$. Consequently, $c(v_1v_2)$ is uniquely determined by the label $\ell(v_1)$ and $c(v_kv_1)$. In general, consider an edge v_jv_{j+1} where $2 \leq j \leq k$. If we proceed from v_kv_1 to v_jv_{j+1} along the path $(v_k, v_1, v_2, \ldots, v_j, v_{j+1})$, then the color $c(v_jv_{j+1})$ of the edge v_jv_{j+1} is

$$c(v_jv_{j+1}) = c(v_kv_1) + \sum_{v\in S_j} \ell(v) + \sum_{v\in T_j} 2\ell(v) \quad \text{in } \mathbb{Z}_3. \tag{1}$$

Since $\sum_{v\in S_k} \ell(v) + \sum_{v\in T_k} 2\ell(v) = 0$ in $\mathbb{Z}_3$ by Lemma 5.2, it follows that

$$c(v_kv_{k+1}) = c(v_kv_1) + \sum_{v\in S_k} \ell(v) + \sum_{v\in T_k} 2\ell(v) = c(v_kv_1) \text{ in } \mathbb{Z}_3.$$

Thus, we return to the color of v_kv_1. Furthermore, every two adjacent edges of C are colored differently and the type of each vertex x of C is $\ell(x)$.

Next, we proceed from v_kv_1 to v_jv_{j+1} along the path $(v_1, v_k, v_{k-1}, \ldots, v_{j+1}, v_j)$. According to the label $\ell(v_k)$ of v_k and the color $c(v_kv_1)$ of v_kv_1, we assign the two colors $\{1,2,3\} - \{c(v_kv_1)\}$ to the two edges v_kv_{k-1} and v_kw_k incident with v_k such that the type of v_k is $\ell(v_k)$. Thus, if v_kw_k lies exterior to C, then $c(v_kv_{k-1}) = c(v_kv_1) + 2\ell(v_k)$ in $\mathbb{Z}_3$; while if v_kw_k lies interior to C, then $c(v_kv_{k-1}) = c(v_kv_1) + \ell(v_k)$ in $\mathbb{Z}_3$. Consequently, $c(v_kv_{k-1})$ is uniquely determined by the label $\ell(v_1)$ and $c(v_kv_1)$. In general, consider an edge v_jv_{j+1} where $1 \leq j \leq k-1$. If we proceed from v_kv_1 to v_jv_{j+1} along the path $(v_1, v_k, v_{k-1}, \ldots, v_{j+1}, v_j)$, the color $c(v_jv_{j+1})$ of the edge v_jv_{j+1} is

$$c(v_jv_{j+1}) = c(v_kv_1) + \sum_{v\in S_k - S_j} 2\ell(v) + \sum_{v\in T_k - T_j} \ell(v) \quad \text{in } \mathbb{Z}_3. \tag{2}$$

It remains to show that the colors of the edge v_jv_{j+1} described in (1) and (2), respectively, are the same. Since

$$\begin{aligned}\sum_{v\in S_k} \ell(v) &+ \sum_{v\in T_k} 2\ell(v) \\ &= \left(\sum_{v\in S_j} \ell(v) + \sum_{v\in T_j} 2\ell(v)\right) + \left(\sum_{v\in S_k - S_j} \ell(v) + \sum_{v\in T_k - T_j} 2\ell(v)\right)\end{aligned}$$

and $\sum_{v\in S_k}\ell(v)+\sum_{v\in T_k}2\ell(v)=0$ in $\mathbb{Z}_3$ by Lemma 5.2, it follows that

$$\begin{aligned}\sum_{v\in S_j}\ell(v)+\sum_{v\in T_j}2\ell(v)&\\ &=-\sum_{v\in S_k-S_j}\ell(v)-\sum_{v\in T_k-T_j}2\ell(v)\\ &=\sum_{v\in S_k-S_j}2\ell(v)+\sum_{v\in T_k-T_j}\ell(v).\end{aligned}$$

Hence,

$$\begin{aligned}c(v_jv_{j+1})&=c(v_kv_1)+\sum_{v\in S_j}\ell(v)+\sum_{v\in T_j}2\ell(v)\\ &=c(v_kv_1)+\sum_{v\in S_k-S_j}2\ell(v)+\sum_{v\in T_k-T_j}\ell(v).\end{aligned}$$

Therefore, the color $c(v_jv_{j+1})$ of v_jv_{j+1} in (1) obtained by proceeding along the path $(v_k, v_1, v_2, \ldots, v_j, v_{j+1})$ is exactly the same as the color $c(v_jv_{j+1})$ of v_jv_{j+1} in (2) obtained by proceeding along the path $(v_1, v_k, v_{k-1}, \ldots, v_{j+1}, v_j)$. Consequently, with a prescribed color $c(v_kv_1)$ and a zonal labeling ℓ of G, there is a unique proper edge coloring of C with the colors $1, 2, 3$ such that the type of each vertex x of C is $\ell(x)$. □

By Theorem 5.3, if G is a cubic map with a zonal labeling ℓ and some edge of a cycle C of G is assigned a color from $\mathbb{Z}_3=\{1,2,3\}$, then there is a unique proper coloring of the edges of C with three colors such that the type of each vertex of C is the label assigned by ℓ. This is not only true for every cycle of G but for G itself.

Theorem 5.4. *Let G be a cubic map with a zonal labeling ℓ. If an edge of G is assigned a color from $\mathbb{Z}_3=\{1,2,3\}$, then there is a unique proper edge coloring of G with the colors $1, 2, 3$ such that the type of each vertex x of G is $\ell(x)$.*

Proof. Let G be a cubic map with a zonal labeling ℓ and let e be an edge of G, where e is assigned a color from the set $\mathbb{Z}_3=\{1,2,3\}$. Let f be an arbitrary edge of G distinct from e. We show that f is assigned a unique color from the zonal labeling ℓ. Since G is 2-connected, there is a cycle C containing both e and f. By Theorem 5.3, there is a unique color $c(f)$ assigned to f from the cycle C. We now show that for every path in G with initial edge e and terminal edge f, the color assigned to f according to the labels of the vertices of P is also $c(f)$ and so the color of f is uniquely determined.

Suppose that the edges of P are encountered in the orders

$$e = f_1, f_2, \ldots, f_k = f.$$

If all edges of P lie on a cycle, then the color $c(f)$ is uniquely determined by Theorem 5.3. Hence, we may assume that P does not lie on a cycle in G. Since G is a cubic map, it follows that there is a sequence $P^{(1)}, P^{(2)}, \ldots, P^{(t)}$ of subpaths of P (proceeding from e to f) such that the terminal edge of $P^{(i)}$ is the initial edge of $P^{(i+1)}$ for $1 \leq i \leq t-1$ and each $P^{(i)}$ belongs to a cycle $C^{(i)}$ of G for $1 \leq i \leq t$. Let $P^{(1)} = (e = f_1, f_2, \ldots, f_{j_1})$, $P^{(2)} = (f_{j_1}, f_{j_1+1}, \ldots, f_{j_2})$, …, $P^{(t)} = (f_{j_{t-1}}, f_{j_{t-1}+1}, \ldots, f_{j_t} = f)$. Applying Theorem 5.3 to the cycle $C^{(1)}$, we see that the color $c(e) = c(f_1)$ and the labels of the vertices of $P^{(1)}$ uniquely determine the color $c(f_i)$ of each edge f_i of $P^{(1)}$ for $2 \leq i \leq j_1$. Since $c(f_{j_1})$ is known, it follows that Theorem 5.3 can be applied to the cycle $C^{(2)}$ and so the color $c(f_{j_1})$ and the labels of the vertices of $P^{(2)}$ uniquely determine the color $c(f_i))$ of each edge f_i of $P^{(2)}$ for $j_1 + 1 \leq i \leq j_2$. Continuing in this manner, we see that the color $c(f_i)$ of each edge f_i of $P^{(t)}$ is uniquely determined for $j_{t-1} + 1 \leq i \leq j_t$. Hence, the colors of all edges of P are uniquely determined, as is the edge f. $\square$

Theorem 5.4 gives us the following result.

Corollary 5.5. *Every zonal cubic map has a proper edge coloring with three colors.*

6 In closing

Cubic maps have been encountered in the study of graph theory as far back as the 19th century. In 1884, Tait made a conjecture dealing with 3-connected cubic planar graphs (a subset of the cubic maps).

Tait's conjecture *Every* 3*-connected cubic planar graph is Hamiltonian.*

This conjecture turned out to be false. In 1946, Tutte [13] gave an example of a 3-connected cubic planar graph of order 46 that is not Hamiltonian.

(Later, counterexamples for this conjecture of smaller order were discovered.) Had Tait's Conjecture turned out to be true (as Tait believed), then this would have been significant for there would be an immediate corollary that there is a proper edge coloring of every 3-connected cubic planar graph that uses three colors. Then the following known theorem could be applied (see [5]).

Theorem 6.1. *There is a proper edge coloring of every cubic map with three colors if and only if there is a proper edge coloring of every* 3*-connected cubic planar graph.*

Therefore, if Tait's Conjecture had been true, then it would have followed by Theorem 6.1 that there is a a proper edge coloring of every cubic map with three colors. From this fact, the following 1880 result of Tait [12] could be applied.

Tait's theorem *There is a proper edge coloring of a cubic map M with three colors if and only if the regions of M can be colored with four or fewer colors so that every two adjacent regions are colored differently.*

Consequently, had Tait's Conjecture been correct, this would have resulted in a proof of the Four Color Theorem.

As we saw, there is a proper edge coloring of a cubic map M with three colors if and only if M is zonal (Theorem 5.1 and Corollary 5.5). Since we know that there is a proper edge coloring of any cubic map with three colors, it follows that every cubic map is zonal. Furthermore, if we know that all cubic maps are zonal, then it follows that the regions of all cubic maps can be properly colored with four colors (by Tait's Theorem). That is, showing that all cubic maps are zonal results in establishing the Four Color Theorem. However, the reason that we know all cubic maps are zonal (Theorem 4.1) is because of the truth of the Four Color Theorem. If this fact could be established (without the aid of the Four Color Theorem), then this would result in an alternative proof of the Four Color Theorem.

Acknowledgment

We thank the anonymous referee for both the careful reading and suggestions.

References

[1] K. Appel and W. Haken, Every planar map is four colorable, *Bull. Amer. Math. Soc.* **82** (1976), 711–712.

[2] A. Bowling and P. Zhang, Absolutely and conditionally zonal graphs *Electron. J. Math.* **4** (2022), 1–11.

[3] G. Chartrand, C. Egan and P. Zhang, *How to Label a Graph*, Springer, New York, 2019.

[4] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz and F. Saba, Irregular networks, *Congr. Numer.* **64** (1988), 197–210.

[5] s G. Chartrand and P. Zhang, *Chromatic Graph Theory, Second Edition*, Taylor & Francis/CRC Press, Boca Raton, 2020.

[6] M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours, *J. Combin. Theory Ser. B* **91** (2004), 151–157.

[7] M. Kalkowski, M. Karoński and F. Pfender, Vertex-coloring edge-weightings: Towards the 1-2-3-conjecture, *J. Combin. Theory Ser. B* **100** (2010), 347–349.

[8] S.W. Golomb, How to number a graph, *Graph Theory and Computing*, Academic Press, New York (1972), 23–37.

[9] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **5** (1998), Dynamic Survey 6, 43 pp.

[10] R.L. Graham and N.J.A. Sloane, On additive bases and harmonious graphs *SIAM J. Alg. Disc. Math.*, **1** (1980), 382–404.

[11] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs*, Gordon and Breach, New York; Dunod, Paris, 349–355, 1967

[12] P.G. Tait, Remarks on the colouring of maps, *Proc. Royal Soc. Edinburgh* **10** (1880), 729.

[13] W.T. Tutte, On Hamiltonian circuits, *J. London Math. Soc.* **21** (1946), 98–101.

[14] H. Whitney, Congruent graphs and the connectivity of graphs, *Am. J. Math.* **54** (1932), 150–168.

[15] R. Wilson, *Four Colors Suffice: How the Map Problem Was Solved*, Princeton University Press, Princeton, NJ, 2002.

Gary Chartrand
Department of Mathematics,
Western Michigan University, Kalamazoo, Michigan, USA
gary.chartrand@wmich.edu

Cooroo Egan
Melbourne, Australia
cooroo@ozlawyer.com

Ping Zhang
Department of Mathematics,
Western Michigan University, Kalamazoo, Michigan, USA
ping.zhang@wmich.edu