



Peakless Motzkin paths of bounded height

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Abstract. There was recent interest in Motzkin paths without peaks (peak: up-step followed immediately by down-step); additional results about this interesting family are worked out. The new results are the enumeration of such paths that live in a strip $[0..l]$ and as consequence the asymptotics of the average height, which is given by $2 \cdot 5^{-1/4} \sqrt{\pi n}$. Methods include the kernel method and singularity analysis of generating functions.

1 Introduction

Motzkin paths are cousins of the more famous Dyck paths. They appear first in [7]. In the encyclopedia [12] they are sequence A001006, with many references given. They consist of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$ and horizontal (flat) steps $F = (1, 0)$. Slightly different notations are also in use. They start at the origin and must never go below the x -axis. Usually one requires the path to end on the x -axis as well, but in our context one uses the term *Motzkin path* also for paths that end on a different level. Figure 1 shows all Motzkin paths of 4 steps (i.e., length 4).

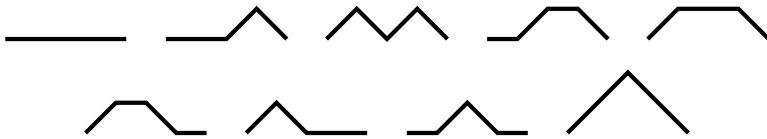


Figure 1.1: All 9 Motzkin path of 4 steps (length 4).

An important concept is the *height* of a path. It is the maximal y -coordinate when scanning the path (from left to right, say). For the paths in Figure 1, the heights are (in this order) 0, 1, 1, 1, 1, 1, 1, 1, 2. The average height of all

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Figure 1.2: Motzkin paths with peaks indicated.

Motzkin paths of length n was computed in an early paper of the present writer [9].

Recently, I learned from the paper [2] that there is interest in *peakless Motzkin paths*. A peak in a Motzkin path is a sequence of an up-step followed immediately by a down-step. Figure 1.2 indicates all peaks in the list of Motzkin paths of length 4. The enumerating sequence is A004148 in [12], where one can find several references; one recent paper about the subject is [3]. A general discussion about forbidden patterns, concentrating on analytic aspects, is in [1].

The paths without peaks are called peakless, and there are four of them, as shown in Figure 1.3.



Figure 1.3: Peakless Motzkin paths of length 4.

2 A warmup: Enumeration of peakless Motzkin paths via the kernel method

We will show how to efficiently deal with peakless Motzkin paths using generating functions. We set up a graph (automaton) that describes the possible transitions. We have an infinity of states but draw only finitely many of them. Each one corresponds to a generating function, namely counting paths that are peakless Motzkin paths and end in this state. Recurrences may be written, according to the last step such a path did in order to reach the current state. There is the variable z to take care of the number of steps, mostly implicit, and, more importantly, a variable u , such that u^j represents paths ending on level j . The origin is the special state where every path starts. In automata theory language, this is the starting state. Loops correspond to level steps.

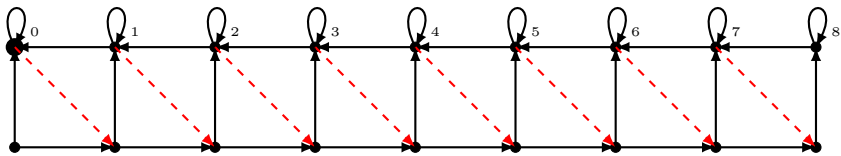


Figure 2.1: Graph (automaton) to recognize peakless Motzkin paths; only the first few states are shown. Starting at the origin and ending at nodes labelled 0 corresponds to Motzkin paths and ending at a node labelled k to a path that ends at level k .

We will use the following generating functions enumerating paths related to peakless Motzkin paths: $[z^n]f_i(z)$ is the number of peakless paths ending at state i in the top layer (Figure 2.1), $[z^n]g_i(z)$ is the number of peakless paths ending at state i in the bottom layer; for convenience, we mostly write f_i and g_i . The second layer serves the purpose of avoiding peaks: a chain of up-steps can only be interrupted by a flat step, not a down-step. The following recursions¹ can be read off the automaton, by considering the last step separately:

$$\begin{aligned} f_0 &= 1 + zf_0 + zf_1 + zg_0, \\ f_i &= zf_i + zf_{i+1} + zg_i, \quad i \geq 1, \\ g_0 &= 0, \\ g_{i+1} &= zf_i + zg_i, \quad i \geq 0. \end{aligned}$$

To solve this system, one introduces double generating functions

$$F(u, z) = \sum_{i \geq 0} u^i f_i(z) \quad \text{and} \quad G(u, z) = \sum_{i \geq 0} u^i g_i(z).$$

Again, for convenience, we mostly write $F(u)$ and $G(u)$. By summing the recurrences, we find

$$\begin{aligned} F(u) &= 1 + zF(u) + \frac{z}{u}(F(u) - F(0)) + zG(u), \\ G(u) &= zuF(u) + zuG(u) \Rightarrow G(u) = \frac{zuF(u)}{1 - zu}. \end{aligned}$$

Eliminating one function, we are left to solve (note that $F(0) = f_0$)

$$F(u) = 1 + zF(u) + \frac{z}{u}(F(u) - F(0)) + \frac{z^2 u F(u)}{1 - zu}.$$

¹The terms recursion and recurrence are used synonymously.

Rewriting the functional equation, we find

$$F(u) = \frac{(-u + zF(0))(1 - zu)}{zu^2 + (z - z^2 - 1)u + z} = \frac{(-u + zF(0))(1 - zu)}{z(u - s_1)(u - s_2)}$$

with

$$s_1 = \frac{1 - z + z^2 + \sqrt{(1 + z + z^2)(1 - 3z + z^2)}}{2z} \quad (1)$$

and

$$s_2 = \frac{1 - z + z^2 - \sqrt{(1 + z + z^2)(1 - 3z + z^2)}}{2z}. \quad (2)$$

Note that $s_1 s_2 = 1$. Plugging $u = 0$ into that equation does not help, but one of the factors from the denominator can be cancelled. This is a (simple) instance of the kernel method. Some twenty years ago, I collected various related examples [10], and many more just recently [11].

Since $s_2 = z + z^2 + \dots$, the factor $(u - s_2)$ must cancel, since $1/(u - s_2)$ would not have a power series expansion around u, z close to zero. Performing the cancellation, we get

$$F(u) = \frac{zs_2 - 1 + zu - z^2F(0)}{z(u - s_1)} \quad \text{and} \quad F(0) = \frac{zs_2 - 1 - z^2F(0)}{-zs_1}.$$

Solving,

$$\begin{aligned} F(0) = f_0 = \frac{s_2}{z} &= \frac{1 - z + z^2 - \sqrt{(1 + z + z^2)(1 - 3z + z^2)}}{2z^2} \\ &= 1 + z + z^2 + 2z^3 + 4z^4 + 8z^5 + 17z^6 + \dots \end{aligned}$$

We are mostly interested in

$$H(u) := F(u) + G(u) = \frac{-u + zF(0)}{zu + z - z^2u - u + zu^2},$$

although $F(u)$ and $G(u)$ could be computed separately as well. The coefficients of $[u^k]H(u)$ are just enumerating all peakless Motzkin paths ending on level k . Hence

$$H(u) = \frac{-u + zF(0)}{zu + z - z^2u - u + zu^2} = \frac{-1}{z(u - s_1)}.$$

Expanding and noting that $s_1 = 1/s_2$, we find

$$[u^k]H(u) = \frac{s_2^{k+1}}{z}.$$

This could be seen as well by a canonical decomposition of a peakless Motzkin path according to the last return to the x -axis. While such direct combinatorial arguments are charming, the machinery of generating functions is arguably more powerful. Bijective proofs are challenging and interesting but also cumbersome in places.

The denominator of $H(u)$ has a special significance; if we write $h_k = [u^k]H(u)$, the recursion for these quantities can be read off from the denominator:

$$zh_k + (z - z^2 - 1)h_{k-1} + zh_{k-2} = 0, \quad (3)$$

which can be checked directly as well by inserting $h_k = s_2^{k+1}/z$ and simplifying.

The sequence enumerating peakless Motzkin paths is A004148 in [12]. If we denote the relevant quantities $m(n) = [z^n]s_2/z$, then the software Gfun, implemented in Maple, produces the recursion

$$\begin{aligned} nm(n) - (2n + 3)m(n + 1) - (n + 3)m(n + 2) \\ - (2n + 9)m(n + 3) + (n + 6)m(n + 4) = 0, \end{aligned}$$

with initial values $m(0) = 1, m(1) = 1, m(2) = 1, m(3) = 2$.

A second order linear recursion with constant coefficients is driven by the characteristic equation and its two roots. Not surprisingly, they are s_1 and s_2 .

For completeness, we mention the asymptotics of the coefficients $m(n)$ of

$$\frac{s_2}{z} = \frac{1 - z + z^2 - \sqrt{(1 + z + z^2)(1 - 3z + z^2)}}{2z^2}.$$

This is a standard application of *singularity analysis of generating functions*, as described in [5]. First, we must consider the closest singularity to the origin. The candidates are the solutions of $(1 + z + z^2)(1 - 3z + z^2) = 0$. There are two complex solutions of absolute value 1, which are irrelevant, and then $\frac{3 \pm \sqrt{5}}{2}$. The relevant value is $\varrho = \frac{3 - \sqrt{5}}{2}$; note that $1/\varrho = \frac{3 + \sqrt{5}}{2}$, which is the square of the golden ratio $\phi = \frac{1 + \sqrt{5}}{2}$ from the Fibonacci fame.

The local expansion around $z \sim \varrho$ looks like

$$\frac{s_2}{z} \sim \frac{1}{\varrho} - \frac{5^{1/4}}{\varrho} \sqrt{1 - \frac{z}{\varrho}},$$

and following the principles of singularity analysis we might translate this to the coefficients

$$[z^n] \frac{s_2}{z} \sim \frac{5^{1/4} \varrho^{-n-1}}{2\sqrt{\pi n^{3/2}}}.$$

3 Peakless Motzkin paths of bounded height

We fix a parameter $\ell \geq 0$ and postulate that $[u^j]H(u) = 0$ for $j > \ell$. This means that states $\ell + 1, \ell + 2, \dots$ (on both layers) can never be reached.

The recursion is, compare (3),

$$zh_{k+1} + (z - z^2 - 1)h_k + zh_{k-1} = 0,$$

but one has to be careful about the initial values so that this recursion only holds true for $k \geq 1$. Furthermore, $(z - z^2 - 1)h_0 + zh_1 = -1$ and $(z - 1)h_0 = -1$.

The system is then best written as a matrix equation:

$$\begin{pmatrix} z - z^2 - 1 & z & 0 & 0 & \dots \\ z & z - z^2 - 1 & z & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ & & & z & z - z^2 - 1 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_\ell \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let \mathcal{D}_ℓ be the determinant of the $(\ell + 1) \times (\ell + 1)$ matrix

$$\begin{pmatrix} z - z^2 - 1 & z & 0 & 0 & \dots \\ z & z - z^2 - 1 & z & 0 & 0 & \dots \\ 0 & z & z - z^2 - 1 & z & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ & & & z & z - z^2 - 1 \end{pmatrix}.$$

We have these initial values:

$$\mathcal{D}_0 = -z^2 + z - 1 \quad \text{and} \quad \mathcal{D}_1 = (z - z^2 - 1)^2 - z^2 = (1 - z)^2(1 + z^2).$$

The solution is, by standard methods,

$$\mathcal{D}_\ell = -\frac{1}{W}(-zs_1)^{\ell+2} + \frac{1}{W}(-zs_2)^{\ell+2},$$

where we use the abbreviation $W = \sqrt{(1+z+z^2)(1-3z+z^2)}$.

The quantity h_0 , describing all paths, restricted as described, can, by Cramer's rule, be written as

$$\begin{aligned} h_0 &= \frac{-\mathcal{D}_{\ell-1}}{\mathcal{D}_\ell} = -\frac{-\frac{1}{W}(-zs_1)^{\ell+1} + \frac{1}{W}(-zs_2)^{\ell+1}}{-\frac{1}{W}(-zs_1)^{\ell+2} + \frac{1}{W}(-zs_2)^{\ell+2}} \\ &= \frac{1 - (-s_1)^{\ell+1} + (-s_2)^{\ell+1}}{z \frac{(-s_1)^{\ell+2} - (-s_2)^{\ell+2}}{s_1^{\ell+2} - s_2^{\ell+2}}} = \frac{1}{z} \frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}}. \end{aligned}$$

This formula works only for $\ell \geq 1$, since $\mathcal{D}_0 = z - z^2 - 1$ but it should be $z - 1$, since we need $h_0 = \frac{1}{1-z}$. This can be repaired: We define

$$\mathcal{D}_\ell^* = \mathcal{D}_\ell \quad \text{for } \ell \geq 1 \quad \text{and} \quad \mathcal{D}_0^* = z - 1.$$

Then

$$\mathcal{D}_\ell^* = z(1-z)\mathcal{D}_\ell - (1+z^2)(1-z)^2\mathcal{D}_{\ell-1}.$$

In the limit $\ell \rightarrow \infty$, $h_0 = \frac{1}{zs_1} = \frac{s_2}{z}$. This is, as we have seen already, the generating function of peakless Motzkin paths without boundary. The other functions h_i could be computed by Cramer's rule as well, but we concentrate only on the paths that return to the origin and are bounded by ℓ . They live in the strip $[0..\ell]$. At this stage, we drop the '0' from the notation and make the ' ℓ ' explicit by writing $A_{n,\ell}$. We summarize the results:

Theorem 3.1. *The number of peakless Motzkin paths of length n (returning to the x -axis), bounded by ℓ , is given by*

$$A_{n,\ell} = [z^n] \frac{1}{z} \frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}},$$

with the functions s_1 and s_2 given in (1) and (2). The formula is only correct for $\ell \geq 1$. If $\ell = 0$, there is only one such path of length of n , namely consisting of flat steps only; the generating function must then be replaced by $\frac{1}{1-z}$.

We will use the notations ($\ell \geq 1$)

$$A_\ell(z) = \frac{1}{z} \frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}} \quad \text{and} \quad A_\infty(z) = \frac{s_2}{z}.$$

As can be checked directly (best with a computer), there is the recursion of the continued fraction type

$$A_\ell = \frac{1}{1 - z + z^2 - z^2 A_{\ell-1}},$$

and thus

$$A_1 = \frac{1}{1 - z + z^2 - \frac{z^2}{1 - z + z^2}}, \quad A_2 = \frac{1}{1 - z + z^2 - \frac{z^2}{1 - z + z^2 - \frac{z^2}{1 - z + z^2}}}$$

and so on. As discussed, the formula for $A_0 = \frac{1}{1-z}$ is slightly different. In the limit,

$$A_\infty = \frac{1}{1 - z + z^2 - z^2 A_\infty},$$

and this quadratic equation can be solved directly, leading to the explicit form discussed earlier.

Continued fraction expansions are very common in the context of generating functions of lattice paths of bounded height; the first example is (perhaps) [4].

Here is a general remark how to obtain such continued fraction expansions. Start from

$$\mathcal{D}_\ell = (z - z^2 - 1)\mathcal{D}_{\ell-1} + (1 + z^2)(1 - z)^2 \mathcal{D}_{\ell-2}$$

and thus

$$\frac{\mathcal{D}_\ell}{\mathcal{D}_{\ell-1}} = (z - z^2 - 1) + (1 + z^2)(1 - z)^2 \frac{\mathcal{D}_{\ell-2}}{\mathcal{D}_{\ell-1}}.$$

With $\tau_\ell = \frac{\mathcal{D}_{\ell-1}}{\mathcal{D}_\ell}$, this is

$$\tau_\ell = \frac{1}{(z - z^2 - 1) + (1 + z^2)(1 - z)^2 \tau_{\ell-1}}.$$

This can be iterated and leads to recursions of the continued fraction type.

The total generating function A_∞ has a pretty expansion as well, viz.

$$A_\infty = 1 + \frac{z}{1 - \frac{z}{1 - \frac{z}{1 - \frac{z^3}{1 - \frac{z}{1 - \frac{z}{1 - \frac{z^3}{1 - \frac{z}{1 - \frac{z^3}{1 - \dots}}}}}}}}}$$

This can be checked directly. I am wondering whether this might have an easy combinatorial interpretation.

Now we consider the *average height* of peakless Motzkin paths of length n , assuming that all of them are equally likely. As always, when enumerating the average height, the relevant formula is

$$\frac{[z^n] \sum_{\ell \geq 0} (A_\infty(z) - A_\ell(z))}{[z^n] A_\infty(z)}.$$

The difference $A_\infty(z) - A_\ell(z)$ enumerates paths of height $> \ell$, or

$$\frac{s_2}{z} - \frac{1}{z} \frac{s_1^{\ell+1} - s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}} = \frac{1}{z} \frac{(1 - s_2^2) s_2^{\ell+1}}{s_1^{\ell+2} - s_2^{\ell+2}} = \frac{W}{z^2} \frac{s_2^{\ell+2}}{s_1^{\ell+2} - s_2^{\ell+2}}.$$

This must be expanded around $z = \varrho$; we decrease ℓ by one, since then we enumerate paths of height $\geq \ell$. Since $s_2 \sim 1$, we find the approximation

$$\frac{W}{\varrho^2} \frac{1}{s_1^{\ell+1} - 1} = \frac{W}{\varrho^2} \frac{s_2^{\ell+1}}{1 - s_2^{\ell+1}} = \frac{W}{\varrho^2} \sum_{k \geq 1} s_2^{(\ell+1)k}.$$

A local expansion yields

$$\frac{W}{\varrho^2} \sim \frac{2 \cdot 5^{1/4}}{\varrho} \left(1 - \frac{z}{\varrho}\right)^{1/2}.$$

From our earlier computation,

$$s_2 \sim 1 - 5^{1/4} \left(1 - \frac{z}{\varrho}\right)^{1/2}.$$

From [6] we conclude that

$$\sum_{k \geq 1} \frac{s_2^k}{1 - s_2^k} \sim -\frac{\log(1 - s_2)}{1 - s_2},$$

as $s_2 \rightarrow 1$. Our paper [6] has many more technical details about a similar scenario. Putting both expansions together (we don't care about a missing term in the sum as we only want to work out the leading term),

$$\begin{aligned} \frac{W}{\varrho^2} \sum_{k, \ell \geq 1} s_2^{(\ell+1)k} &\sim \frac{2 \cdot 5^{1/4}}{\varrho} \left(1 - \frac{z}{\varrho}\right)^{1/2} \cdot \frac{-\log(1 - s_2)}{1 - s_2} \\ &\sim \frac{2 \cdot 5^{1/4}}{\varrho} \left(1 - \frac{z}{\varrho}\right)^{1/2} \cdot \frac{-\log\left(5^{1/4}\left(1 - \frac{z}{\varrho}\right)^{1/2}\right)}{5^{1/4}\left(1 - \frac{z}{\varrho}\right)^{1/2}} \\ &\sim -\frac{2}{\varrho} \log\left(5^{1/4}\left(1 - \frac{z}{\varrho}\right)^{1/2}\right) \sim -\frac{2}{\varrho} \log\left(1 - \frac{z}{\varrho}\right)^{1/2} \\ &\sim -\frac{1}{\varrho} \log\left(1 - \frac{z}{\varrho}\right). \end{aligned}$$

By singularity analysis (transfer theorem) we find that

$$[z^n] \frac{W}{\varrho^2} \sum_{k, \ell \geq 1} s_2^{(\ell+1)k} \sim -[z^n] \frac{1}{\varrho} \log\left(1 - \frac{z}{\varrho}\right) \sim \frac{\varrho^{-n-1}}{n}.$$

As discussed before, the total number of peakless Motzkin paths of length n is asymptotic to

$$[z^n] \frac{s_2}{z} \sim \frac{5^{1/4} \varrho^{-n-1}}{2\sqrt{\pi n} 3^{3/2}},$$

and for the average height we have to consider the quotient of the last two expressions, which is

$$\frac{\varrho^{-n-1}}{n} \frac{2\sqrt{\pi n} 3^{3/2}}{5^{1/4} \varrho^{-n-1}} = \frac{2\sqrt{\pi n}}{5^{1/4}}.$$

The numerical constant $2/5^{1/4} = 1.337480610$. This can be compared with the average height of *all* Motzkin paths of length n , which is asymptotic to $\sqrt{\frac{\pi n}{3}}$, see [8]; $3^{-1/2} = 0.5773502693$.

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