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Further results on the metric dimension of circulant graphs with 2 or 3 generators

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Abstract. Let $G = (V, E)$ be a connected graph. A subset $W = \{v_1, \dots, v_k\} \subseteq V$ is called a resolving set of G if $r(u \,|\, W) \neq r(v \,|\, W)$ for every $u, v \in V$, $u \neq v$, where $r(u \,|\, W) = \big(d(u, v_1), \dots, d(u, v_k)\big)$ and $d(u, v_i)$ is the graph distance between u and v_i, $1 \leq i \leq k$. The cardinality of a smallest possible resolving set of G is called the metric dimension of G, which is denoted by $\beta(G)$. Extensive research has been conducted on the metric dimension of various graph classes, including circulant graphs, which exhibit interesting properties. Let n, m and $a_1, a_2, \dots, a_m$ be positive integers such that $1 \leq a_1 < a_2 < \cdots < a_m \leq \frac{n}{2}$. A circulant graph $C_n(a_1, a_2, \dots, a_m)$ consists of vertices $v_0, v_1, \dots, v_{n-1}$ and edges $v_i v_{i+a_j}$ for $i = 0, \dots, n-1$ and $j = 1, \dots, m$ where the indices are modulo n. In this paper, we sharpen an upper bound of $\beta\big(C_n(1, 3)\big)$ found by Javaid et al. [9], and we disprove an exact value of $\beta\big(C_n(1, 2, 4)\big)$ given by Imran and Bokhary [8], both by providing a smaller resolving set, respectively.

1 Introduction

The metric dimension problem of graphs was introduced by Slater [14] as *location number* of connected graphs. The main idea of this problem is to find a subset W of vertices in a connected graph G such that all vertices in the graph have distinct representations according to their distances to the vertices of W. This subset W is called the *locating set* of G. Independently, Harary and Melter [5] also introduced this concept as *metric dimension*, and the set W is called a *resolving set*. The concept of metric dimension of graphs was used in various applications such as robot navigation [11] and chemistry [2]. See [12, 15] for excellent surveys on this topic.

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All graphs considered in this paper are finite, simple, and undirected. Let $G = (V, E)$ be a connected graph and $W = \{v_1, \ldots, v_k\} \subseteq V$. If necessary, then we write $V = V(G)$ and $E = E(G)$. The *distance* between two vertices u and v in G, denoted by $d_G(u, v)$ or $d(u, v)$ if the graph is clear, is the length of the shortest path in G connecting u and v. The *diameter* of G is defined as $\operatorname{diam}(G) := \max\{d(u, v) : u, v \in V\}$. For a vertex $u \in V$ and $1 \le d \le \operatorname{diam}(G)$, we define $N_d(u) := \{v \in V : d(u, v) = d\}$. For a vertex $v \in V$, the *coordinate* of v with respect to W is the k-tuple $r(v \,|\, W) = \big(d(v, v_1), d(v, v_2), \ldots, d(v, v_k)\big)$. If $r(u \,|\, W) \neq r(v \,|\, W)$ for every $u, v \in V$, $u \neq v$, then W is called a *resolving set* of G. If W has the smallest possible size, then W is called a *metric basis* of G and $|W|$ is called the *metric dimension* of G, which is denoted by $\beta(G)$. Some characterizations of graphs with metric dimension 1 or 2, which we use later in this paper, are as follows.

Theorem 1.1 (Chartrand et al. [2] and Khuller et al. [11]). *A graph G has $\beta(G) = 1$ if and only if G is a path.*

Theorem 1.2 (Khuller et al. [11]). *Let G be a graph with $\beta(G) = 2$ and let $\{a, b\} \subset V(G)$ be a metric basis of G. Then, the degrees of a and b are at most* 3.

Many results on the metric dimension have been found for various graph classes including circulant graphs, which exhibit interesting properties. Let n, m and $a_1, a_2, \ldots, a_m$ be positive integers such that $1 \le a_1 < a_2 < \cdots < a_m \le \frac{n}{2}$. A *circulant graph* $C_n(a_1, a_2, \ldots, a_m)$ consists of vertices $v_0, v_1, \ldots, v_{n-1}$ and edges $v_i v_{i+a_j}$ for $0 \le i \le n-1$ and $1 \le j \le m$ where the indices are modulo n. The numbers $a_1, a_2, \ldots, a_m$ are called *generators*. Note that $C_n(a_1, a_2, \ldots, a_m)$ is $2m$-regular if all of its generators are less than $\frac{n}{2}$, and $(2m-1)$-regular if $\frac{n}{2}$ is a generator.

In 2008, Javaid et al. [10] found the values of $\beta\big(C_n(1, 2)\big)$ both the exact values and upper bounds for certain cases. On the other hand, in 2012, Imran et al. [6] found the exact values and upper bounds for $\beta\big(C_n(1, 2, 3)\big)$. All these results were improved later in 2018 by Borchert and Gosselin [1] where they provide the exact values of $\beta\big(C_n(1, 2)\big)$ and $\beta\big(C_n(1, 2, 3)\big)$ completely.

Furthermore, results on the metric dimensions of circulant graphs with 2 or 3 generators were investigated by, for instance, Javaid et al. [9] ($C_n(1, 3)$), Salman et al. [13] ($C_n(1, \frac{n}{2})$), Imran and Bokhary [8] ($C_n(1, 2, 4)$), and Imran et al. [7] ($C_n(1, 2, 5)$). General results for $C_n(1, 2, \ldots, k)$ were found by Grigorius et al. [4] where the upper bounds were improved later by Chau and Gosselin [3] and Vetrík et al. [16].

In this paper, we improve some of the aforementioned results. Javaid et al. [9] proved that $4 \le \beta\big(C_n(1,3)\big) \le 6$ for $n \ge 8$ and $n \equiv 2 \pmod 6$. Here, we sharpen this upper bound by proving that $\beta\big(C_n(1,3)\big) \le 5$ for $n \ge 22$ and $n \equiv 2 \pmod 6$. Furthermore, Imran and Bokhary [8] claimed that for $n \ge 13$, $\beta\big(C_n(1,2,4)\big) = 4$. However, we find that for certain conditions, this is not the case. Indeed, we prove that $\beta\big(C_n(1,2,4)\big) = 3$ for $n \ge 22$ and $n \equiv 6 \pmod 8$.

2 Main results

Theorem 2.1. *For $n \ge 20$ with $n \equiv 2 \pmod 6$, we have $\beta\big(C_n(1,3)\big) \le 5$.*

Proof. Let $G = C_n(1,3)$, $W_0 = \{0,1,4,5\}$, and define $W = W_0 \cup \left\{\frac{n}{2}+1\right\}$. We intend to show that W is a resolving set of G by proving that if two vertices i and j satisfy $r(i \,|\, W_0) = r(j \,|\, W_0)$, then necessarily $d(i, \frac{n}{2}+1) \neq d(j, \frac{n}{2}+1)$. To do this, we show that it suffices to consider the vertices $3 \le i \le \frac{n}{2}+2$, due to the symmetries of the graph.

Consider placing a mirror between vertices 2 and 3, dividing G into two symmetric halves. Define the map $\pi\colon i \mapsto 5 - i \mod n$, which is an automorphism of G that reflects each vertex across the mirror. One can verify that $\pi(0) = 5$, $\pi(5) = 0$, $\pi(1) = 4$, and $\pi(4) = 1$, so $\pi(W_0) = W_0$. Since π preserves distances, we have $d(i,j) = d\big(\pi(i), \pi(j)\big)$ for all $i, j \in V(G)$, and thus $r(i \,|\, W_0)$ is the reverse of $r\big(\pi(i) \,|\, W_0\big)$.

Direct computation yields

$$r(i \,|\, W_0) = \begin{cases} (1,2,1,2), & i = 3; \\ (2,1,0,1), & i = 4; \\ (3,2,1,0), & i = 5; \\ (2,3,2,1), & i = 6; \\ \frac{1}{6}(n-2, n+4, n-2, n-8), & i = \frac{n}{2}+1; \\ \frac{1}{6}(n+4, n-2, n+4, n-2), & i = \frac{n}{2}+2; \end{cases}$$

and for $7 \le i \le \frac{n}{2}$, we have

$$r(i \,|\, W_0) = \frac{1}{3}\begin{cases} (i, i+3, i, i-3), & i \equiv 0 \pmod 3; \\ (i+2, i-1, i-4, i-1), & i \equiv 1 \pmod 3; \\ (i+4, i+1, i-2, i-5), & i \equiv 2 \pmod 3. \end{cases}$$

To show that no two distinct vertices $i \neq j$ in the range $3 \leq i, j \leq \frac{n}{2} + 2$ satisfy $r(i \,|\, W_0) = r(j \,|\, W_0)$, except for the pair $\left\{\frac{n}{2} - 1, \frac{n}{2} + 1\right\}$, we consider the sum $s(i) := \sum_{w \in W_0} d(i, w)$ of the entries in $r(i \,|\, W_0)$. If two vertices have the same coordinates, then their sums must also be equal; conversely, if their sums differ, then so do their coordinates. Therefore, it suffices to examine only those pairs with equal sums.

Using the above formulas, we compute the following values:

$$s(3) = 6, \qquad s(4) = 4, \qquad s(5) = 6,$$
$$s\left(\tfrac{n}{2} - 1\right) = \tfrac{1}{3}(2n - 4), \qquad s\left(\tfrac{n}{2}\right) = \tfrac{1}{3}(2n - 4),$$
$$s\left(\tfrac{n}{2} + 1\right) = \tfrac{1}{3}(2n - 4), \qquad s\left(\tfrac{n}{2} + 2\right) = \tfrac{1}{3}(2n + 2),$$

and for $2 \leq k \leq \frac{n-8}{6}$, we have $s(3k) = s(3k+1) = 4k$ and $s(3k+2) = 4k+2$. We thus only need to examine the following candidate pairs with equal sums: $\{3, 5\}$, $\{\frac{n}{2} - 1, \frac{n}{2}\}$, $\{\frac{n}{2}, \frac{n}{2} + 1\}$, $\{\frac{n}{2} - 1, \frac{n}{2} + 1\}$, and $\{3k, 3k+1\}$ for $2 \leq k \leq \frac{n-8}{6}$. A direct inspection of the coordinate vectors confirms that all these pairs have distinct coordinates except for $\left\{\frac{n}{2} - 1, \frac{n}{2} + 1\right\}$.

Furthermore, none of the coordinates of $3 \leq i \leq \frac{n}{2} + 2$ are reverses of each other. This implies, by the symmetry of G, that it suffices to consider only the vertices in the range $3 \leq i \leq \frac{n}{2} + 2$. That is, all remaining vertices in G also have distinct coordinates with respect to W_0 except for the pair $\{\frac{n}{2} + 4, \frac{n}{2} + 6\} = \pi\big(\{\frac{n}{2} - 1, \frac{n}{2} + 1\}\big)$.

Finally, upon considering the additional vertex $\frac{n}{2} + 1 \in W$, we observe

$$d(\tfrac{n}{2} - 1, \tfrac{n}{2} + 1) = 2 \neq 0 = d(\tfrac{n}{2} + 1, \tfrac{n}{2} + 1), \text{ and}$$
$$d(\tfrac{n}{2} + 4, \tfrac{n}{2} + 1) = 1 \neq 3 = d(\tfrac{n}{2} + 6, \tfrac{n}{2} + 1),$$

and thus $r(\frac{n}{2} - 1 \,|\, W) \neq r(\frac{n}{2} + 1 \,|\, W)$ and $r(\frac{n}{2} + 4 \,|\, W) \neq r(\frac{n}{2} + 6 \,|\, W)$. Therefore, all vertices in G have distinct coordinates with respect to W, and hence W is a resolving set of G. □

Theorem 2.2. *For $n \geq 22$ with $n \equiv 6 \pmod 8$, $\beta\big(C_n(1, 2, 4)\big) = 3$.*

Proof. First, we prove the upper bound. Let $G = C_n(1, 2, 4)$ and $W = \{0, 1, n - 5\} \subseteq V(G)$. We intend to prove that W is a resolving set of $C_n(1, 2, 4)$ with $|W| = 3$, and hence $\beta(G) \leq 3$. Similar to the proof of Theorem 2.1, we show that if two vertices i and j satisfy $r\big(i \,|\, \{0, 1\}\big) = r\big(j \,|\, \{0, 1\}\big)$, then

$d(i,n-5) \neq d(j,n-5)$. Let i and j be two distinct vertices in G. The distance between i and j in G is given by the formula

$$\begin{aligned} d(i,j) &= \left\lfloor \frac{\delta_{ij}}{4} \right\rfloor + \left\lfloor \frac{\delta_{ij} \bmod 4}{2} \right\rfloor + \big((\delta_{i,j} \bmod 4) \bmod 2\big) \\ &= \tfrac{1}{4}\Big(\delta_{i,j} + \delta_{i,j} \bmod 4 + 2\big((\delta_{i,j} \bmod 4) \bmod 2\big)\Big) \end{aligned}$$

where $\delta_{i,j} := \min\big\{|i-j|, n-|i-j|\big\}$. Note that $D := \operatorname{diam}(G) = \frac{1}{8}(n+10)$.

We note that the set $\bigcap_{w\in\{0,1\}} N_{d_w}(w)$ is the set of all vertices $x \in V(G)$ having the coordinate $r\big(x \,|\, \{0,1\}\big) = (d_0, d_1)$. We are only interested in the case where $\big|\bigcap_{w\in\{0,1\}} N_{d_w}(w)\big| \geq 2$. Observe that

$$N_{d_0}(0) = \begin{cases} \{1,2,4,n-4,n-2,n-1\}, & d_0 = 1; \\ \{4d_0-5, 4d_0-3, 4d_0-2, 4d_0, n-4d_0, \\ \quad n-4d_0+2, n-4d_0+3, n-4d_0+5\}, & 2 \leq d_0 \leq D-2; \\ \{4d_0-5, 4d_0-3, 4d_0-2, n-4d_0+2, \\ \quad n-4d_0+3, n-4d_0+5\}, & d_0 = D-1; \\ \{\frac{n}{2}\}, & d_0 = D. \end{cases}$$

Since $\big|\bigcap_{w\in\{0,1\}} N_{d_w}(w)\big| \geq 2$, we only need to consider three cases of d_0.

Case 1. If $d_0 = 1$, then we have

$$\bigcap_{w\in\{0,1\}} N_{d_w}(w) = \begin{cases} \{1\}, & d_1 = 0; \\ \{2, n-1\}, & d_1 = 1; \\ \{4, n-4, n-2\}, & d_1 = 2. \end{cases}$$

Since $\big|\bigcap_{w\in\{0,1\}} N_{d_w}(w)\big| \geq 2$, we must have $d_1 = 1$ or $d_1 = 2$. It is easy to verify that, for $d_1 = 1$, we have $d(2,n-5) = 3$ and $d(n-1,n-5) = 1$. Furthermore, for $d_1 = 2$, we have $d(4,n-5) = 3$, $d(n-4,n-5) = 1$, and $d(n-2,n-5) = 2$. Therefore, $r(i \,|\, W) \neq r(j \,|\, W)$ for all $\{i,j\}$ given that $d(i,0) = d(j,0) = 1$.

Case 2. If $d_0 = D-1$, then we have

$$\bigcap_{w\in\{0,1\}} N_{d_w}(w) = \begin{cases} \{4d_0-5, 4d_0-3, n-4d_0+5\}, & d_1 = d_0 - 1; \\ \{4d_0-2, n-4d_0+3\}, & d_1 = d_0; \\ \{n-4d_0+2\}, & d_1 = d_0 + 1. \end{cases}$$

Since $\big|\bigcap_{w\in\{0,1\}} N_{d_w}(w)\big| \geq 2$, we must have $d_1 = d_0 - 1$ or $d_1 = d_0$. For $d_1 = d_0 - 1$, we have $d(4d_0-5, n-5) = d_0$, $d(4d_0-3, n-5) = d_0 - 1$, and

$d(n-4d_0+5, n-5) = d_0 - 2$. For $d_1 = d_0$, we have $d(4d_0-2, n-5) = d_0$ and $d(n-4d_0+3, n-5) = d_0 - 2$. Therefore, $r(i \mid W) \neq r(j \mid W)$ for all $\{i, j\}$ given that $d(i, 0) = d(j, 0) = D - 1$.

Case 3. If $2 \leq d_0 \leq D - 2$, then we have

$$\bigcap_{w \in \{0,1\}} N_{d_w}(w) = \begin{cases} \{4d_0-5, 4d_0-3, n-4d_0+5\}, & d_1 = d_0 - 1; \\ \{4d_0-2, n-4d_0+3\}, & d_1 = d_0; \\ \{4d_0, n-4d_0, n-4d_0+2\}, & d_1 = d_0 + 1. \end{cases}$$

For $d_1 = d_0 - 1$, we have $d(4d_0-5, n-5) = d_0$, $d(4d_0-3, n-5) = d_0 + 1$, and

$$d(n-4d_0+5, n-5) = \begin{cases} 1, & n = 22; \\ d_0 - 2, & n > 22. \end{cases}$$

Thus, $r(i|W) \neq r(j|W)$ for all $\{i, j\}$ given that $r\big(i \mid \{0,1\}\big) = r\big(j \mid \{0,1\}\big) = (d_0, d_0 - 1)$. For $d_1 = d_0$, we have $d(4d_0-2, n-5) = d_0+2$ and $d(n-4d_0+3, n-5) = d_0 - 2$. Therefore, $r(i \mid W) \neq r(j \mid W)$ for all $\{i, j\}$ given that $r(i \mid \{0,1\}) = r(j \mid \{0,1\}) = (d_0, d_0)$. Lastly, for $d_1 = d_0 + 1$, we have

$$d(4d_0, n-5) = \begin{cases} d_0 + 2, & 2 \leq d_0 \leq D - 3; \\ \frac{1}{4}(n - 4d_0 - 2), & d_0 = D - 2; \end{cases}$$

$d(n-4d_0, n-5) = d_0$; and $d(n-4d_0+2, n-5) = d_0 - 1$. It is easy to verify that all the distances obtained are distinct; thus, $r(i \mid W) \neq r(j \mid W)$ for all $\{i, j\}$ given that $r(i \mid \{0,1\}) = r(j \mid \{0,1\}) = (d_0, d_0 + 1)$. Therefore, $r(i \mid W) \neq r(j \mid W)$ for all $\{i, j\}$ given that $d(i, 0) = d(j, 0) = d_0$ for $2 \leq d_0 \leq D - 2$.

From these three cases, we conclude that $r(i \mid W) \neq r(j \mid W)$ for all $i, j \in V(G)$, $i \neq j$. Therefore, W is a resolving set of G with $|W| = 3$, and $\beta(G) \leq 3$.

Now, we prove the lower bound. Since G is not a path, then $\beta(G) \geq 2$ by Theorem 1.1. Furthermore, assume that $\beta(G) = 2$. Observe that every vertex in G has degree $6 > 3$, and thus any set $W = \{a, b\} \subset V(G)$ cannot be a metric basis of G by Theorem 1.2, which is a contradiction. Therefore, $\beta(G) \geq 3$, and the proof is complete. □

3 Conclusion

In this paper, we have improved a result by Javaid et al. [9] by proving that $\beta\big(C_n(1,3)\big) \leq 5$ for $n \geq 22$ and $n \equiv 2 \pmod 6$, which sharpened the existing upper bound for $\beta\big(C_n(1,3)\big)$. We also disproved the exact value of $\beta\big(C_n(1,2,4)\big)$ given by Imran and Bokhary [8] by proving a lower exact value, that is, $\beta\big(C_n(1,2,4)\big) = 3$ for $n \geq 22$ and $n \equiv 6 \pmod 8$.

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