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A 2-person game decomposing 2-manifolds

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This paper is dedicated to Lee Leonard (1951–2025), my dear friend of more than 60 years. Many years ago, Lee conceived of the crazy idea of playing games on 2-manifolds and worked out much of the analysis. With persistence he eventually convinced me to join the project.—D. R. Berman

Abstract. Two players play a game by alternately splitting a surface of a compact 2-manifold along a simple closed curve that is not null-homotopic and attaching disks to the resulting boundary; the last player who can move wins. Starting from an orientable surface, the G-series is $01\overline{20}$ according to increasing genus. Starting from a nonorientable surface, the G-series is $012\overline{4603}$ according to increasing genus. Nim addition determines the G-values of the remaining compact 2-manifolds.

1 Introduction

In this paper we introduce a two-player game played on a two-dimensional topological manifold and give a winning strategy. We begin by reviewing necessary material from combinatorial game theory and topology.

Combinatorial games have a rich history in recreational mathematics and number theory. The definitive book is *Winning Ways for Your Mathematical Plays* by Berlekamp, Conway, and Guy [1], which contains theory, history, and numerous games that are played and analyzed. Our game involves two players who move alternately; the last one who can move wins. The game terminates after a finite number of moves. The game is impartial, meaning the same moves are available to each player. The quintessential such game is *nim*.

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Nim is played with a finite set of heaps, each containing a finite number of counters. On a move, a player chooses a heap and removes one or more counters. The last player who can move wins. A winning strategy is found using the nim sum of the heap sizes. The *nim sum* of a and b is denoted $a \oplus b$ and is found by writing the numbers in binary and adding with no carries. This is the same as the binary sum using XOR instead of ordinary addition. Equivalently, write the numbers as sums of powers of 2 and remove any repeated powers of 2. The sum of the remaining powers of 2 is the nim sum. For example, to find $29 \oplus 14$, use binary to get $11101 \oplus 1110 = 10011$, or 19. Using powers of 2, we have $29 = 16+8+4+1$ and $14 = 8+4+2$. So $29 \oplus 14 = 16 + 2 + 1 = 19$. This operation is associative and commutative, so we can nim sum multiple heap sizes.

The winning strategy for nim is based on the nim sum of the heap sizes. First player wins if the sum is positive and second player wins if the sum is zero. We now summarize the proof. For a single null heap—that is, one with no counters—first player has no moves and second player wins by default. The null heap is assigned nim sum zero. We now proceed inductively. If the sum is positive, then first player should be able to remove one or more counters from some heap to ensure that the new game has nim sum zero. If the sum is zero, then for every move first player makes, the sum will be positive. Thus first and second players have been reversed and we continue this process until the null heap is left. Note: the nim sum of a single heap of size n is assigned the nim sum of the heap and the null heap, or n.

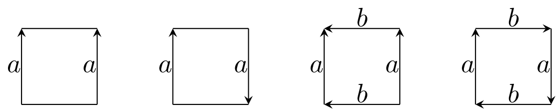
The implementation of the winning strategy is based on the powers of 2 representation of the nim sum. Express each heap size as a sum of powers of 2 and form the multiset consisting of all these powers. If the nim sum is zero, then each power of 2 must occur an even number of times in the multiset. If the nim sum is positive, then some power of 2 occurs an odd number of times in the multiset. The nim sum is calculated by removing as many pairs of like powers of 2 as possible leaving only single powers of 2. The sum of the remaining powers of 2 is the nim sum. Part 1 of the strategy: Suppose the sum is positive. Take the largest power of 2 in the multiset that occurs an odd number of times and take any heap whose size contains that power. Consider all the powers of 2 that occur an odd number of times in the multiset. If the power of 2 is part of the chosen heap, then subtract it; otherwise, add it in. The resulting new game has a multiset with all powers of 2 occurring an even number of times, so the nim sum is zero. Part 2 of the strategy: If the sum is zero, all powers are paired in the multiset, so removing counters from a single heap will leave some power unpaired, and the nim sum will be positive.

Here is an example of the winning strategy for first player. Consider the nim game with heap sizes 50, 53, 30, and 14. Note $50 = 32 + 16 + 2$, $53 = 32 + 16 + 4 + 1$, $30 = 16 + 8 + 4 + 2$, and $14 = 8 + 4 + 2$. The multiset is $[32, 32, 16, 16, 16, 8, 8, 4, 4, 4, 2, 2, 2, 1]$. The powers 16, 4, 2, and 1 occur an odd number of times and 32 and 8 occur an even number of times. Therefore, the nim sum of the heaps is $16 + 4 + 2 + 1 = 23$, a win for first player, who has three possible winning moves corresponding to heaps containing the power 16 in the sum. Possibility 1: Take the heap of size 50, subtract $16 + 2$, and add $4 + 1$ to get the new heap of size 37. Possibility 2: Take the heap of size 53, subtract $16 + 4 + 1$, and add 2 to get the new heap of size 34. Possibility 3: Take the heap of size 30, subtract $16 + 4 + 2$, and add 1 to get the new heap of size 9. With each possible move, the nim sum of the new game is zero as needed.

A winning strategy for any two-player, combinatorial, impartial, terminating game can be determined using the Sprague-Grundy theory, which assigns a non-negative integer, called the *Grundy number*, to each game position. For short we call the Grundy number the *G-value*. The G-value of terminal positions—that is, with no moves possible—is 0. These are losses for the current player and wins for the next player. Proceed inductively. The G-value of a given game position is the minimal excluded integer, or "mex", of the G-value of the game positions after each possible move. For example, suppose a player has three possible moves from a game position, and their G-values are calculated to be 1, 3, and 5. Then $\text{mex}\{1, 3, 5\} = 0$, a win for the next player. For any move, the next player will have a game of positive G-value and can win. Suppose a player has four possible moves from a game position, and their G-values are 0, 1, 3, and 5. Then $\text{mex}\{0, 1, 3, 5\} = 2$, a win for the current player, who plays the move with G-value 0. Finally, if a player can move into any of several disjoint subgames, then the G-value of the position is the nim sum of the G-values of the subgames, and the analysis is similar to nim. A winning strategy can now be stated. If the initial game's G-value is positive, then the first player wins by moving to a position with G-value 0. If the initial game G-value is 0, then the second player wins because every move results in a position of positive G-value. For example, if a player can choose either of the hypothetical positions above, then the player wins as the nim sum of 0 and 2 is 2, which is positive. The player moves from the second subgame with a move of G-value 0.

Suppose a game is indexed by the nonnegative integers, meaning there is a game instance corresponding to each integer. The *G-series* is the sequence of G-values $\{G(0), G(1), \ldots\}$. For example, consider the game that begins with a single heap. In a move a player may remove one counter from a heap or split a heap into two heaps. We show in Theorem 2.3 that the G-series is $01202020\ldots$, or $01\overline{20}$ for short. We next turn our attention to topology.

Classification of compact surfaces—that is, compact and connected 2-manifolds—is well known and is stated below. We follow the exposition given by Massey [6], and we have used [4, 5, 7, 8] as well. A compact 2-manifold is composed of a finite disjoint collection of compact surfaces. The *connected sum* of two surfaces is formed by removing a disk from each surface and joining the surfaces along the boundary. The basic compact surfaces are the *sphere*, *torus*, and *projective plane*. The sphere and torus are orientable and the projective plane nonorientable. We also use the *annulus* and *Möbius band*, which are bordered surfaces. Standard constructions of these are shown below. Edges of each square with the same label are identified in the direction of the arrow.



annulus | Möbius band | torus | projective plane

We need the observation that a *Möbius band* with a disk attached to the edge is homeomorphic to the projective plane. A beautiful pictorial proof is given in [2]. The disk is called a *cross-cap*.

Theorem 1.1. *Every compact surface is homeomorphic to either a sphere, a connected sum of g tori, or a connected sum of g projective planes, where g is the genus. The genus of a sphere is* 0.

We denote the connected sum of g tori by og, the orientable surface of genus g, and the connected sum of g projective planes by ng, the nonorientable surface of genus g. For convenience we write $o0 = n0$ for the sphere. The genus is related to the Euler characteristic, which is not needed in our presentation. The proof of the classification theorem in Massey uses the following simplification.

Theorem 1.2. *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*

We make use of the following corollary.

Corollary 1.3. *If $g = a+b$ with $b > 0$ and a is even, then the connected sum of g projective planes is homeomorphic to the connected sum of $a/2$ tori and b projective planes.*

We are now ready to introduce our whimsical game. While not playable in practice due to the difficulty in visualizing nonorientable surfaces, it is nevertheless interesting to study and analyze.

2 Manifold decomposition game

Assume a 2-manifold is given and is composed of compact surfaces. Two players play a game by alternately forming a proper decomposition of a compact surface. A proper decomposition starts with an essential, simple, closed curve, denoted J. Essential means the curve is not null-homotopic, or equivalently does not bound a disk in the manifold (see Massey [6, Chapter 2, Lemma 8.1]). The simple closed curve is expanded into a tubular—that is, regular—neighborhood. The neighborhood is either an annulus or a Möbius band. Remove the interior of the neighborhood leaving the boundary, two circles for the annulus and one circle for the Möbius band. The decomposition—that is, move—is completed by capping off the circles. Each move results in either a single surface or a pair of surfaces. The game begins with either og, the orientable surface of genus g, or ng, the nonorientable surface of genus g. The game ends with a collection of spheres, because there are no essential, simple, closed curves on a sphere.

Theorem 2.1. *For genus g, the following decompositions are always available and comprise all possible proper decompositions of a compact surface. These are the moves in the manifold decomposition game:*

(a) $og \to o(g-1)$ $\quad g-1 \geq 0$,

(b) $og \to (oa, ob)$ $\quad a, b > 0$ *and* $a+b = g$,

(c) $ng \to n(g-1)$ $\quad g-1 \geq 0$,

(d) $ng \to o((g-1)/2)$ $\quad g-1 \geq 0$ *and even*,

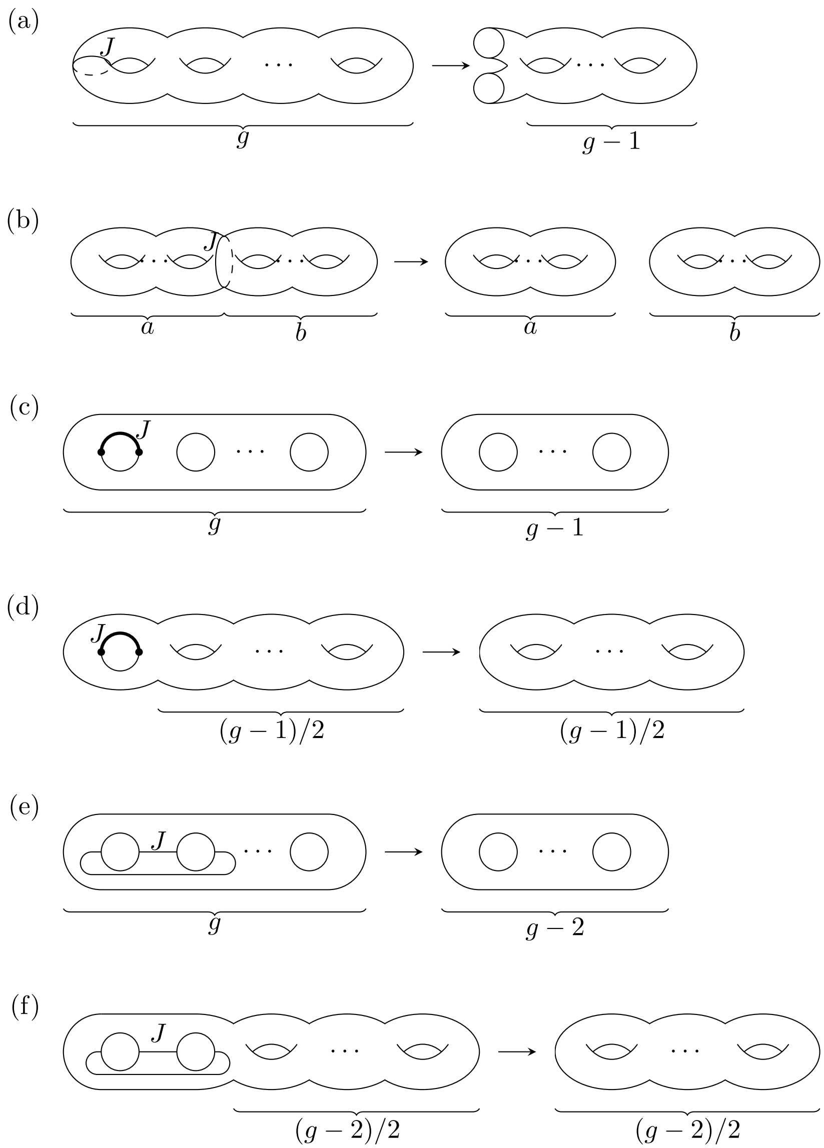
(e) $ng \to n(g-2)$ $\quad g-2 \geq 0$,

(f) $ng \to o((g-2)/2)$ $\quad g-2 \geq 0$ *and even*,

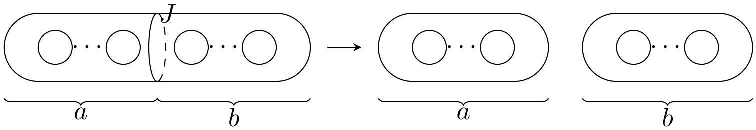
(g) $ng \to (na, nb)$ $\quad a, b > 0$ *and* $a+b = g$,

(h) $ng \to (o(a/2), nb)$ $\quad a, b > 0$, *a even, and* $a+b = g$.

Proof. The cases are visualized below. Cases d, f, and h make use of e and Corollary 1.3. In cases c and d, the curve J is identified antipodally. The tubular neighborhood is a Möbius band. The projective plane—that is, crosscap—is composed of the Möbius band and the boundary. When the Möbius band is removed and the boundary capped, the projective plane disappears. In cases e and f, the curve J involves two crosscaps. The tubular neighborhood joins the two Möbius bands forming an annulus, and capping the boundaries with disks removes both projective planes.



(g)



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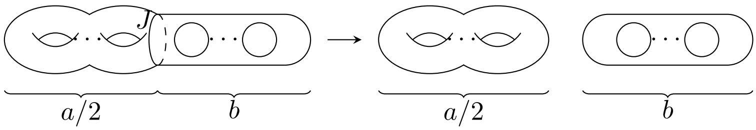

□

Corollary 2.2. *Starting from og or ng, the manifold decomposition game ends after at most 2g moves. The smallest number of moves is k for n(2k) and k + 1 for n(2k + 1).*

Proof. All moves decrease the genus sum except cases (b) and (g). In these two cases, the genus sum stays the same, and the number of components that are not spheres increases by 1. Hence, the games with the most moves would use these two cases repeatedly for g moves until there are g components that are not spheres left, each of genus 1. After g more moves using either case (a) or (c), there will be g surfaces left of genus 0—that is, spheres. There are no moves from a sphere, so the game ends.

Starting from $n(2k)$, use case (e) repeatedly to get a sphere after k moves. Starting from $n(2k+1)$, use case (d) once to get ok, and then use case (a) repeatedly to get a sphere after k more moves. □

We next show that starting from an orientable surface the G-series is $01\overline{20}$ according to increasing genus, and starting from a nonorientable surface the G-series is $012\overline{4603}$ according to increasing genus. Nim addition determines the G-values of the remaining compact 2-manifolds. The G-series is composed of Grundy numbers, which we call G-values for short. A line over digits indicate a repeating block.

Theorem 2.3. *The G-series for og is* $01\overline{20}$.

Proof. From Theorem 2.1 there are only two possible moves from og for $g > 0$: either $og \to o(g-1)$ or $og \to (oa, ob)$ for $a, b > 0$ and $g = a + b$. This game is equivalent to the octal game 4.3 from Guy and Smith [3].

Their game starts with a heap of g counters, and in each move a player may split a heap into two heaps or remove one counter from a heap. For completeness, we provide a proof of the claim.

The proof is by induction on g. Initial G-values are computed below:

$$G(o0) = G(\emptyset) = 0, \qquad G(o1) = \text{mex}\{G(o0)\} = \text{mex}\{0\} = 1,$$

$$\begin{aligned} G(o2) &= \text{mex}\{G(o1), G(o1, o1)\} = \text{mex}\{1, G(o1) \oplus G(o1)\} \\ &= \text{mex}\{1, 1 \oplus 1\} = \text{mex}\{1, 0\} = 2. \end{aligned}$$

Assume $g > 2$ and G-values for smaller genus have been established.

Case 1: g is even. Then a, b are both even or both odd. If both are even, then by hypothesis $G(a)$ and $G(b)$ are both 2, and their nim sum is 0. If both are odd, then either one of a or b is 1, or both are larger than 1. Then the G-value is either the nim sum of 1 and 0 or the nim sum of 0 and 0. Thus $G(g) = \text{mex}\{Go(g-1), 0, 1, 0\} = \text{mex}\{0, 0, 1, 0\} = 2$.

Case 2: g is odd. Then a and b have opposite parity with G-values 0 or 1, and 2, and nim sum 2 or 3. Thus $G(g) = \text{mex}\{Go(g-1), 2, 3\} = \text{mex}\{2, 2, 3\} = 0$. □

Theorem 2.4. *The G-series for ng is* $012\overline{4603}$.

Proof. Table 2.1 shows the computation of G-values of ng for $g = 0, \ldots, 14$. For each g, each move from ng is given along with the move G-value. The G-value of ng, i.e. $G(ng)$, is the minimal excluded non-negative integer, or "mex", of these move G-values. We next show by induction that, for g sufficiently large, the set of G-values for moves from ng is the same as the set of G-values for moves from $n(g+4)$. It then follows that $G(ng) = G(n(g+4))$.

The possible moves from ng and $n(g+4)$ correspond to cases (c), (d), ..., (h) in Theorem 2.1. We consider each case in turn.

Case (c): $ng \to n(g-1)$ and $n(g+4) \to n(g+3)$. If the G-values of the right sides are equal, then the G-values of the left sides must be equal. The proof proceeds by induction.

Case (e): $ng \to n(g-2)$ and $n(g+4) \to n(g+2)$. Similar to case (c).

Case (d): $ng \to o((g-1)/2)$ if $g-1 \geq 0$ and even, and $n(g+4) \to o((g+3)/2)$. Similar to above but using the previous theorem for the G-series for $o(g)$.

Case (f): $ng \to o((g-2)/2)$ if $g-2 \geq 0$ and even, and $n(g+4) \to o((g+2)/2)$. Similar to case (d).

Case (g). We list the moves in two tables, for $g = 2k$ and for $g = 2k+1$.

$$\begin{array}{rll} n(2k) \to & (n1, n(2k-1)), & \ldots, A = (n(k-2), n(k+2)), \\ & B = (n(k-1), n(k+1)), & (nk, nk). \\ n(2k+4) \to & (n1, n(2k+3)), & \ldots, (n(k-2), n(k+6)), \\ & (n(k-1), n(k+5)), & (nk, n(k+4)), \\ & B' = (n(k+1), n(k+3)), & A' = (n(k+2), n(k+2)). \end{array}$$

The moves from $n(2k)$ correspond inductively to moves from $n(2k+4)$ in the order listed. The two additional moves from $n(2k+4)$ labeled A' and B' correspond to moves from $n(2k)$ labeled A and B.

$$\begin{array}{rll} n(2k+1) \to & (n1, n(2k)), & \ldots, (n(k-2), n(k+3)), \\ & C = (n(k-1), n(k+2)), & D = (nk, n(k+1). \\ n(2k+5) \to & (n1, n(2k+4)), & \ldots, (n(k-2), n(k+7)), \\ & (n(k-1), n(k+6)), & (nk, n(k+5)), \\ & D' = (n(k+1), n(k+4)), & C' = (n(k+2), n(k+3)). \end{array}$$

Same as above except C corresponds to C' and D to D'.

Case (h): We list the moves in two tables, for $g = 2k$ and for $g = 2k+1$.

$$\begin{array}{rll} n(2k) \to & (o1, n(2k-2)), & \ldots, A = (o(k-3), n6)), \\ & B = (o(k-2), n4), & C = (o(k-1), n2). \\ n(2k+4) \to & (o1, n(2k+2)), & \ldots, (o(k-3), n10), \\ & (o(k-2), n8), & A' = (o(k-1), n6), \\ & B' = (ok, n4), & C' = (o(k+1), n2). \end{array}$$

The moves from $n(2k)$ correspond inductively to moves from $n(2k+4)$ in the order listed, with the exception of the rule labeled C, as $G(n2) \neq G(6)$. The two additional moves from $n(2k+4)$ labeled B' and C' correspond to moves from $n(2k)$ labeled B and C. Note that for A and A' to correspond we must have $k > 4$ to use the periodicity of the G-series of $o(g)$. So $2k+4 > 12$.

$$\begin{array}{rll} n(2k+1) \to & (o1, n(2k-1)), & \ldots, D = (o(k-2), n5)), \\ & E = (o(k-1), n3)), & F = (ok, n1), \\ n(2k+5) \to & (o1, n(2k+3)), & \ldots, (o(k-1), n7), \\ & D' = (ok, n5), & E' = (o(k+1), n3), \\ & F' = (o(k+2), n1). & \end{array}$$

Same as above except D corresponds to D', E to E', and F to F'.

Table 2.1 shows the computation of G-values of ng for $g = 0, \dots, 14$. We see that $G(ng) = G(n(g+4))$ for $3 \le g \le 10$. Assume $g \ge 9$ and that, for all g' with $12 \le g' \le g+3$, the G-value of ng' is given. Then for each case above, the set of G-values for moves from ng is the same as the set of G-values for moves from $n(g+4)$. Thus, the set of G-values for all moves from ng is the same as the set of G-values for all moves from $n(g+4)$, and so $G(ng) = G(n(g+4))$. □

Table 2.1: Calculating initial G-values of ng.

g	Moves from ng and G-values			$G(ng)$
0		$\emptyset$		0
1	$n0$, 0			1
2	$n1$, 1	$n0$, 0	$(n1, n1)$, 0	2
3	$n2$, 2	$n1$, 1	$o1$, 1	4
	$(n1, n2)$, 3	$(o1, n1)$, 0		
4	$n3$, 4	$n2$, 2	$o1$, 1	6
	$(n1, n3)$, 5	$(n2, n2)$, 0	$(o1, n2)$, 3	
5	$n4$, 6	$n3$, 4	$o2$, 2	0
	$(n1, n4)$, 7	$(n2, n3)$, 6	$(o1, n3)$, 5	
	$(o2, n1)$, 3			
6	$n5$, 0	$n4$, 6	$o2$, 2	3
	$(n1, n5)$, 1	$(n2, n4)$, 4	$(n3, n3)$, 0	
	$(o1, n4)$, 7	$(o2, n2)$, 0		
7	$n6$, 3	$n5$, 0	$o3$, 0	4
	$(n1, n6)$, 2	$(n2, n5)$, 2	$(n3, n4)$, 2	
	$(o1, n5)$, 1	$(o2, n3)$, 6	$(o3, n1)$, 1	
8	$n7$, 4	$n6$, 3	$o3$, 0	6
	$(n1, n7)$, 5	$(n2, n6)$, 1	$(n3, n5)$, 4	
	$(n4, n4)$, 0	$(o1, n6)$, 2	$(o2, n4)$, 4	
	$(o3, n2)$, 2			
9	$n8$, 6	$n7$, 4	$o4$, 2	0
	$(n1, n8)$, 7	$(n2, n7)$, 6	$(n3, n6)$, 7	
	$(n4, n5)$, 6	$(o1, n7)$, 5	$(o2, n5)$, 2	

Continued on next page

Table 2.1 – continued from previous page.

g	Moves from ng and G-values			$G(ng)$
	$(o3, n3)$, 4	$(o4, n1)$, 3		
10	$n9$, 0	$n8$, 6	$o4$, 2	3
	$(n1, n9)$, 1	$(n2, n8)$, 4	$(n3, n7)$, 0	
	$(n4, n6)$, 5	$(n5, n5)$, 0	$(o1, n8)$, 7	
	$(o2, n6)$, 1	$(o3, n4)$, 6	$(o4, n2)$, 0	
11	$n10$, 3	$n9$, 0	$o5$, 0	4
	$(n1, n10)$, 2	$(n2, n9)$, 2	$(n3, n8)$, 2	
	$(n4, n7)$, 2	$(n5, n6)$, 3	$(o1, n9)$, 1	
	$(o2, n7)$, 6	$(o3, n5)$, 0	$(o4, n3)$, 6	
	$(o5, n1)$, 1			
12	$n11$, 4	$n10$, 3	$o5$, 0	6
	$(n1, n11)$, 5	$(n2, n10)$, 1	$(n3, n9)$, 4	
	$(n4, n8)$, 0	$(n5, n7)$, 4	$(n6, n6)$, 0	
	$(o1, n10)$, 2	$(o2, n8)$, 4	$(o3, n6)$, 3	
	$(o4, n4)$, 4	$(o5, n2)$, 2		
13	$n12$, 6	$n11$, 4	$o6$, 2	0
	$(n1, n12)$, 7	$(n2, n11)$, 6	$(n3, n10)$, 7	
	$(n4, n9)$, 6	$(n5, n8)$, 6	$(n6n7)$, 7	
	$(o1, n11)$, 5	$(o2, n9)$, 2	$(o3, n7)$, 4	
	$(o4, n5)$, 2	$(o5, n3)$, 4	$(o6, n1)$, 3	
14	$n13$, 0	$n12$, 6	$o6$, 2	3
	$(n1, n13)$, 1	$(n2, n12)$, 4	$(n3, n11)$, 0	
	$(n4, n10)$, 5	$(n5, n9)$, 0	$(n6, n8)$, 5	
	$(n7, n7)$, 0	$(o1, n12)$, 7	$(o2, n10)$, 1	
	$(o3, n8)$, 6	$(o4, n6)$, 1	$(o5, n4)$, 6	
	$(o6, n2)$, 0			

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