



The deleted edge numbers $de(P_m, K_{1,n})$ and $de(K_{1,3}, K_{m,n})$

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Abstract. In 2019, Budden and Dejonge defined the k -deleted Ramsey number, where $k \in \mathbb{N}$, and the deleted edge number in order to study the destruction of Ramsey property by the removal of edges incident at any fixed vertex. Given simple graphs G and H , the k -deleted Ramsey number $D_k(G, H)$ is the least $n \in \mathbb{N}$ such that every 2-edge-colouring of $K_n - E(K_{1,k})$, $k \leq n - 1$, contains a copy of G in colour 1 or a copy of H in colour 2. The deleted edge number $de(G, H)$ is defined as the least $k \in \mathbb{N}$ such that $D_{k-1}(G, H) < D_k(G, H)$. In this paper, we give bounds to $de(G, H)$, when $(G, H) = (P_m, K_{1,n})$ for certain values of m and n ; and when $(G, H) = (K_{1,3}, K_{m,n})$ for all values of m and n , where $n \geq m \geq 2$. Also, we prove that $de(P_m, K_{1,n}) = R(P_m, K_{1,n}) - 1$, when $n \equiv 1 \pmod{m - 1}$.

1 Introduction

In this paper, we consider only undirected, finite, simple graphs with no isolated vertices. For a graph G , $V(G)$ and $E(G)$ denote its vertex set and the edge set, respectively.

Given graphs G and H , the graph Ramsey number $R(G, H)$ is the smallest positive integer n such that every 2-edge-colouring (E_1, E_2) of the complete graph K_n contains a copy of G in E_1 or a copy of H in E_2 , where E_1 is the subgraph of K_n spanned by the edges of colour 1 and E_2 is the subgraph of K_n spanned by the edges of colour 2. This definition implies the existence of a 2-edge-colouring of K_{n-1} such that there is neither a copy of G in colour 1 nor a copy of H in colour 2. In [3], Cowen proved that $K_{R(K_m, K_n)} - e$, where e is any edge of $K_{R(K_m, K_n)}$, has a 2-edge-colouring such that there is no

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K_m in colour 1 and no K_n in colour 2. This means that, removing an edge from $K_{R(K_m, K_n)}$ destroys the Ramsey property.

Let $k \in \mathbb{N}$. As a generalization of Cowen's work [3], Budden and Dejonge [2] defined the k -deleted Ramsey number $D_k(G, H)$ to be the least $n \in \mathbb{N}$ such that every 2-edge-colouring of $K_n - E(K_{1,k})$, $k \leq n - 1$, contains a colour 1 copy of G or a colour 2 copy of H . By convention, $D_0(G, H) := R(G, H)$. From the definition, one can see that, for $k \leq R(G, H)$, we have $R(G, H) \leq D_k(G, H) \leq R(G, H) + 1$. As both $R(G, H)$ and $D_k(G, H)$ take on values from \mathbb{N} , these inequalities imply, for $k \leq R(G, H)$, that $D_k(G, H)$ is either equal to $R(G, H)$ or $R(G, H) + 1$. Further, for all $k' \leq k$, we have that $D_{k'}(G, H) \leq D_k(G, H)$.

For any pair of graphs G and H , the *deleted edge number* $de(G, H)$ is defined, in [2], as the least $k \in \mathbb{N}$ such that $D_{k-1}(G, H) < D_k(G, H)$. This implies that $de(G, H)$ is the least positive integer k such that $D_{k-1}(G, H) = R(G, H)$ and $D_k(G, H) = R(G, H) + 1$. So, to prove that $de(G, H) \geq p$, it suffices to prove $D_{p-1}(G, H) \leq R(G, H)$ (equivalently, any 2-edge-colouring (E_1, E_2) of $K_{R(G, H)} - E(K_{1, p-1})$ contains a copy of G in colour 1 or a copy of H in colour 2); to prove that $de(G, H) \leq q$, it is enough to show that $D_q(G, H) \geq R(G, H) + 1$ (equivalently, there exists a 2-edge-colouring (F_1, F_2) of $K_{R(G, H)} - E(K_{1, q})$ that contains neither a copy of G in colour 1 nor a copy of H in colour 2). If we remove a vertex with all its incident edges from $K_{R(G, H)}$, then it destroys the Ramsey property. So, we have $de(G, H) \leq R(G, H) - 1$. Hence,

$$1 \leq de(G, H) \leq R(G, H) - 1. \tag{1}$$

Prior to the introduction of $de(G, H)$, Hook and Isaak [7] introduced the *star-critical Ramsey number* $r^*(G, H)$ as the smallest integer k such that every 2-edge-coloring of $K_{R(G, H)} - K_{1, R(G, H) - 1 - k}$ contains a colour 1 copy of G or a colour 2 copy of H .

From the definition of $de(G, H)$ and $r^*(G, H)$, it is easy to observe that

$$r^*(G, H) = R(G, H) - de(G, H).$$

For any positive integer n , $K_{1, n-1}$, T_n , P_n , and C_n denote, respectively, a star, a tree, a path, and a cycle of order n .

In [7], Hook and Isaak computed $r^*(G, H)$ when

$$(G, H) \in \{(T_m, K_n), (nK_2, mK_2), (nK_3, mK_3), (P_n, C_4)\}.$$

Further, several authors have worked on $r^*(G, H)$ (see [5, 6, 8, 10, 11]).

Theorem 1.1 (Hook, [6]). *If $m \geq 3$, then $r^*(P_m, P_3) = 2$.*

Theorem 1.2 (Wang et al., [10]). *For any integers m and n with $m \geq 2n+1$, it holds that $r^*(P_m, K_{1,n}) = m - n - 1$.*

In [2], the authors computed the value of $de(G, H)$ to certain graph pairs (G, H) and also provided bounds to some other pairs.

Theorem 1.3 (Budden and Dejonge, [2]).

- (1) *If $m, n \geq 2$ are both even, then $de(K_{1,m}, K_{1,n}) = 1$.*
- (2) *If $m \geq 1$ or $n \geq 1$ is odd, then $de(K_{1,m}, K_{1,n}) = m + n - 1$.*
- (3) *For $m \geq 4$ and $n \geq 1$, if T_m is a non-star tree, then $D_1(T_m, K_{1,n}) \leq m + n - 1$.*
- (4) *For $m, n \geq 4$, if T_m^* and T_n^* are trees of orders m and n with $\Delta(T_m^*) = m - 2$ and $\Delta(T_n^*) = n - 2$, then $D_1(T_m^*, T_n^*) \leq m + n - 3$.*

In this paper, we investigate $de(G, H)$ when

$$(G, H) \in \{(P_m, K_{1,n}), (K_{1,3}, K_{m,n})\}.$$

In Section 3, for $m, n \geq 3$, we show that

$$\begin{aligned} de(P_m, K_{1,n}) &= m + n - 2 && \text{if } n \equiv 1 \pmod{m-1}, \\ de(P_m, K_{1,n}) &\leq m - 2 && \text{if } n \equiv 0 \text{ or } 2 \pmod{m-1}, \\ de(P_m, K_{1,n}) &> 1 && \text{if } n \leq m \leq 2n, \\ de(P_m, K_{1,n}) &\leq n - 1 && \text{if } n + 1 \leq m \leq 2n - 2, \text{ and} \\ de(P_m, K_{1,n}) &\leq m - n && \text{if } m \in \{2n - 1, 2n\}. \end{aligned}$$

In Section 4, we show, for $n \geq m \geq 2$, that $de(K_{1,3}, K_{m,n}) \leq \lceil \frac{n+1}{m} \rceil + 1$.

2 Notation

Notation not defined here can be found in [1].

For $v \in V(G)$, $d_G(v)$ is the degree of v in G , $N_G(v)$ is the neighbour set of v in G , and $N_G[v] = N_G(v) \cup \{v\}$. The minimum and maximum of the degrees of the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively.

If $T \subseteq V(G)$, the *induced subgraph* $G[T]$ is the subgraph of G whose vertex set is T and whose edge set consists of all edges of G which have both ends in T . For any 2-edge-colouring (E_1, E_2) of G , let E_i denote the set of edges of colour i , and we use $G(E_i)$ to denote the spanning subgraph of G with edge set E_i , $i \in \{1, 2\}$.

Let (E_1, E_2) be a 2-edge-colouring of G . For $X \subseteq V(G)$, we use $G(E_i)[X]$ to denote the set of all colour i edges having both ends in X .

If $X, Y \subseteq V(G)$, then $[X, Y]_G = \{e \in E(G) : e = ab, \text{ where } a \in X \text{ and } b \in Y\}$. When $X = \{a\}$, then $[\{a\}, Y]_G$ is denoted as $[a, Y]_G$. Similar notation also holds for $Y = \{b\}$.

3 Path versus star

From Theorem 1.1, we have $de(P_m, P_3) = de(P_m, K_{1,2}) = m - 2$. So, it is enough to consider $de(P_m, K_{1,n})$ for $m, n \geq 3$.

We list some results of Parsons [9] on $R(P_m, K_{1,n})$.

Theorem 3.1 (Parsons, [9]).

- (1) If $n \equiv 1 \pmod{m-1}$, then $R(P_m, K_{1,n}) = m + n - 1$.
- (2) If $n \leq m \leq 2n - 1$, then $R(P_m, K_{1,n}) = 2n - 1$.
- (3) If $m \geq 2n - 1$, then $R(P_m, K_{1,n}) = m$.
- (4) If $n \equiv 0$ or $2 \pmod{m-1}$, then $R(P_m, K_{1,n}) = m + n - 2$.
- (5) If $n \not\equiv 1 \pmod{m-1}$ and $n \geq (m-3)^2$, then $R(P_m, K_{1,n}) = m + n - 2$.

We use the following lemma in proving some of our results.

Lemma 3.2 (Parsons, [9]). *If $\delta(G) \geq k$ and $v \in V(G)$, then there is a P_{k+1} in G which starts at v .*

Theorem 3.3. *If $m \geq 3$ and $n \equiv 1 \pmod{m-1}$, then $de(P_m, K_{1,n}) = m + n - 2$.*

Proof. From inequality (1), we have $de(P_m, K_{1,n}) \leq R(P_m, K_{1,n}) - 1 = m + n - 2$. To prove the other inequality $de(P_m, K_{1,n}) \geq m + n - 2$, we need to prove that any 2-edge-colouring (E_1, E_2) of $T = K_{m+n-1} - E(K_{1, m+n-3})$

assures a P_m in $T(E_1)$ or a $K_{1,n}$ in $T(E_2)$. Let (E_1, E_2) be an arbitrary 2-edge-colouring of $T = K_{m+n-1} - E(K_{1,m+n-3})$. Then, there exists an edge $e = ab$ of T such that $d_T(a) = 1$ and $d_T(b) = m + n - 2$. If $d_{T(E_2)}(b) \geq n$, then we have a $K_{1,n}$ in $T(E_2)$. Hence assume that $d_{T(E_2)}(b) \leq n - 1$. This implies $d_{T(E_1)}(b) \geq m - 1$. The 2-edge-colouring (E_1, E_2) induces a 2-edge-colouring (E'_1, E'_2) in $T' = T - a$. If $\Delta(T'(E'_2)) \geq n$, then we have a $K_{1,n}$ in $T'(E'_2)$, and so in $T(E_2)$. Hence assume that $\Delta(T'(E'_2)) \leq n - 1$. This implies $\delta(T'(E'_1)) \geq m - 2$, since $|V(T')| = m + n - 2$. Now, by Lemma 3.2, we have a P_{m-1} in $T'(E'_1)$ that starts at b . As $d_{T(E_1)}(b) \geq m - 1$, there exists $w \in V(T) \setminus V(P_{m-1})$ such that $bw \in E_1$. This, along with P_{m-1} , gives a path on m vertices in $T(E_1)$. This completes the proof. \square

Theorem 3.4. *If $n \geq 3$ and $m \geq 2n - 1$, then $de(P_m, K_{1,n}) \leq m - n$.*

Proof. As $R(P_m, K_{1,n}) = m$, we need to find a 2-edge-colouring (F_1, F_2) of $T = K_m - E(K_{1,m-n})$ such that neither $T(F_1)$ contains a P_m nor $T(F_2)$ contains a $K_{1,n}$. Let $a \in V(T)$ be such that $d_T(a) = n - 1$. Let $F_1 = E(T - a)$ and $F_2 = E(T) \setminus F_1$. As $|V(T(F_1))| = m - 1$, we have no P_m in $T(F_1)$. Also, as $T(F_2) \cong K_{1,n-1}$, we have no $K_{1,n}$ in $T(E_2)$. \square

Lemma 3.5. *Any 2-edge-colouring of $K_{2n-1} - e$ contains a P_{2n-1} in colour 1 or a $K_{1,n}$ in colour 2.*

Proof. Contrarily, suppose that there exists a 2-edge-colouring (E_1, E_2) of $T = K_{2n-1} - e$, where $e = ab$, such that neither $T(E_1)$ has a P_{2n-1} nor $T(E_2)$ has a $K_{1,n}$. Consider $T' = T - a$. The 2-edge-colouring (E_1, E_2) of T induces a 2-edge-colouring (E'_1, E'_2) in T' . If $\Delta(T'(E'_2)) \geq n$, then we have a $K_{1,n}$ in $T'(E'_2)$, and so in $T(E_2)$, a contradiction. Hence $\Delta(T'(E'_2)) \leq n - 1$. This implies $\delta(T'(E'_1)) \geq n - 2$, since $|V(T')| = 2n - 2$. Similarly, $d_{T(E_1)}(a) \geq n - 2$. Let $P' = v_1v_2 \dots v_k$ be a longest path in $T'(E'_1)$. As $|V(T')| = 2n - 2$, then $k \leq 2n - 2$. Since $\delta(T'(E'_1)) \geq n - 2$, by Lemma 3.2, we have a P_{n-1} in $T'(E'_1)$. Hence $k \geq n - 1$. Set $Y = V(T') \setminus V(P')$. Note the following:

- (i) $\{\{v_1, v_k\}, Y\}_{T'} \subseteq E'_2$ since P' is a longest path in $T'(E'_1)$.
- (ii) For every $y \in Y$ and $i \in \{1, 2, \dots, k - 1\}$, $yv_i \in E'_2$ or $yv_{i+1} \in E'_2$. (Otherwise, there exists $i \in \{1, 2, \dots, k - 1\}$ such that $yv_i, yv_{i+1} \in E'_1$; then $v_1v_2 \dots v_iv_{i+1}v_{i+2} \dots v_k$ is a P_{k+1} in $T'(E'_1)$, a contradiction to the choice of P' .)

- (iii) For $k \geq n$ and for every $i \in \{2, 3, \dots, k-2\}$ with $v_1 v_{i+1} \in E'_1$, we have $v_k v_i \in E'_2$. (Otherwise, there exists $i \in \{2, 3, \dots, k-2\}$ such that $v_1 v_{i+1}, v_k v_i \in E'_1$. If $n \leq k \leq 2n-3$, then $v_1 v_{i+1} v_{i+2} \dots v_k v_i v_{i-1} v_{i-2} \dots v_1$ is a C_k in $T'(E'_1)$, which along with some edge of E'_1 incident at y , where $y \in Y$, provides us a P_{k+1} in $T'(E'_1)$, a contradiction to the choice of P' . If $k = 2n-2$, then $v_1 v_{i+1} v_{i+2} \dots v_{2n-2} v_i v_{i-1} v_{i-2} \dots v_1$ is a C_{2n-2} in $T'(E'_1)$, which along with some edge of E_1 incident at a provides us a P_{2n-1} in $T(E_1)$, a contradiction.)
- (iv) For $k \geq n$, $v_1 v_k \in E'_2$. (Otherwise, $v_1 v_k \in E'_1$. If $n \leq k \leq 2n-3$, then $v_1 v_2 \dots v_k v_1$ is a C_k in $T'(E'_1)$, which along with some edge of E'_1 incident at y , where $y \in Y$, provides us a P_{k+1} in $T'(E'_1)$, a contradiction to the choice of P' . If $k = 2n-2$, then $v_1 v_2 \dots v_{2n-2} v_1$ is a C_{2n-2} in $T'(E'_1)$, which along with some edge of E_1 incident at a provides us a P_{2n-1} in $T(E_1)$, a contradiction.)

Now, we divide the proof into four cases.

Case 1. $k = 2n - 2$.

Then $|V(P')| = 2n - 2$ and $Y = \emptyset$. By (iv), $v_1 v_{2n-2} \in E'_2$. Among the two ends of P' , at least one end is different from b . By symmetry, assume that $b \neq v_1$. Since $\Delta(T(E_2)) \leq n - 1$, we have $d_{T(E_1)}(v_1) \geq n - 1$. If $v_1 a \in E_1$, then we have a P_{2n-1} in $T(E_1)$, a contradiction. So, $v_1 a \in E_2$, and therefore $d_{T(E_1)}(v_1) = d_{T'(E'_1)}(v_1)$. This implies $N_{T(E_1)}(v_1) \subseteq V(P')$. If $d_{T'(E'_1)}(v_1) \geq n$, then, by (iii) and (iv), $d_{T'(E'_2)}(v_{2n-2}) \geq n$, a contradiction to the fact that $\Delta(T'(E'_2)) \leq n - 1$. So, $d_{T'(E'_1)}(v_1) = n - 1$. This, along with (iii) and (iv), implies $d_{T'(E'_2)}(v_{2n-2}) \geq n - 1$. As $\Delta(T'(E'_2)) \leq n - 1$, we have $d_{T'(E'_2)}(v_{2n-2}) = n - 1$ and so, $d_{T'(E'_1)}(v_{2n-2}) = n - 2$. Now, we claim that $v_{2n-2} = b$. Otherwise, $v_{2n-2} \neq b$, and therefore $v_{2n-2} a \in E_1$ or $v_{2n-2} a \in E_2$. If $v_{2n-2} a \in E_1$ (resp. $v_{2n-2} a \in E_2$), then $v_1 v_2 \dots v_{2n-2} a$ is a P_{2n-1} in $T(E_1)$ (resp. $d_{T(E_2)}(v_{2n-2}) = n$), which is a contradiction. Thus $v_{2n-2} = b$. If there exists $i \in \{1, 2, \dots, 2n - 4\}$ such that $av_i, av_{i+1} \in E_1$, then $v_1 v_2 \dots v_i av_{i+1} v_{i+2} \dots v_{2n-2}$ is a P_{2n-1} in $T(E_1)$, a contradiction. So, for every $i \in \{1, 2, \dots, 2n - 4\}$, $av_i \in E_2$ or $av_{i+1} \in E_2$. This implies that $d_{T(E_2)}(a) \geq n - 1$. As $\Delta(T(E_2)) \leq n - 1$, we have $d_{T(E_2)}(a) = n - 1$, and so $d_{T(E_1)}(a) = n - 2$. Thus

$$N_{T(E_1)}(a) = \{v_2, v_4, v_6, \dots, v_{2n-4}\}$$

and

$$N_{T(E_2)}(a) = \{v_1, v_3, v_5, \dots, v_{2n-3}\}.$$

If there exists $j \in \{3, 5, 7, \dots, 2n-3\}$ such that $v_1v_j \in E'_1$, then $av_{j-1}v_{j-2}v_{j-3} \dots v_1v_jv_{j+1}v_{j+2} \dots v_{2n-2}$ is a P_{2n-1} in $T(E_1)$, a contradiction. So, $v_1v_j \in E'_2$, for every $j \in \{3, 5, 7, \dots, 2n-3\}$, which along with the edges v_1a and v_1v_{2n-2} (both of which are in E_2) gives us a $K_{1,n}$ in $T(E_2)$ with centre v_1 , a contradiction.

Case 2. $k = 2n - 3$.

Then $|Y| = 1$. Let $Y = \{y\}$. Clearly, $N_{T'(E'_1)}(y) \subseteq V(P') \setminus \{v_1, v_{2n-3}\}$. The facts $\delta(T'(E'_1)) \geq n-2$ and (ii) imply $d_{T'(E'_1)}(y) = n-2$. As $yv_1, yv_{2n-3} \in E'_2$, we have $N_{T'(E'_1)}(y) = \{v_2, v_4, v_6, \dots, v_{2n-4}\}$ and $N_{T'(E'_2)}(y) = \{v_1, v_3, v_5, \dots, v_{2n-3}\}$. If $v_1v_j \in E'_1$, where $j \in \{3, 5, 7, \dots, 2n-5\}$, then $yv_{j-1}v_{j-2}v_{j-3} \dots v_1v_jv_{j+1}v_{j+2} \dots v_{2n-3}$ is a P_{2n-2} in $T'(E'_1)$, a contradiction to the choice of P' . So, for every $j \in \{3, 5, 7, \dots, 2n-5\}$, we have $v_1v_j \in E'_2$. Thus

$$\{v_3, v_5, v_7, \dots, v_{2n-5}, v_{2n-3}, y\} \subseteq N_{T'(E'_2)}(v_1).$$

This implies $d_{T'(E'_2)}(v_1) \geq n-1$. As $T'(E'_2)$ is $K_{1,n}$ -free, we have $d_{T'(E'_2)}(v_1) = n-1$; i.e., $N_{T'(E'_2)}(v_1) = \{v_3, v_5, v_7, \dots, v_{2n-5}, v_{2n-3}, y\}$ and $N_{T'(E'_1)}(v_1) = \{v_2, v_4, v_6, \dots, v_{2n-4}\}$. Among the two ends of P' , at least one end is different from b . By symmetry, assume that $b \neq v_1$. If $ay, av_1 \in E_1$, then $yav_1v_2 \dots v_{2n-3}$ is a P_{2n-1} in $T(E_1)$, a contradiction. If $ay \in E_2$ (resp. $av_1 \in E_2$), then we have a $K_{1,n}$ with centre y and pendant vertices in $\{v_1, v_3, v_5, \dots, v_{2n-3}, a\}$ (resp. centre v_1 and pendant vertices in $\{v_3, v_5, v_7, \dots, v_{2n-3}, y, a\}$). In either case, we have a $K_{1,n}$ in $T(E_2)$, a contradiction.

Case 3. $n \leq k \leq 2n - 4$.

Then $2 \leq |Y| \leq n-2$, and so, for every $y \in Y$, $|N_{T'}(y) \cap Y| \leq n-3$, in particular $|N_{T'(E'_1)}(y) \cap Y| \leq n-3$. This, together with $\delta(T'(E'_1)) \geq n-2$, implies that, for every $y \in Y$, $|N_{T'(E'_1)}(y) \cap V(P')| \geq 1$. By (i), $N_{T'(E'_1)}(v_1) \subseteq V(P')$. This, along with (iii) and (iv), implies $|N_{T'(E'_2)}(v_k) \cap V(P')| \geq n-2$. As $|Y| \geq 2$, by (i), we have $d_{T'(E'_2)}(v_k) \geq n$, a contradiction.

Case 4. $k = n - 1$.

Then $|V(P')| = |Y| = n-1$. As $\delta(T'(E'_1)) \geq n-2$, by (i), we have $v_1v_i, v_{n-1}v_i \in E'_1$ for every i , $1 \leq i \leq n-1$. Specifically, $v_1v_{n-1} \in E'_1$. Thus we have the cycle $C' = v_1v_2 \dots v_{n-1}v_1$ on $n-1$ vertices in $T'(E'_1)$. For some $y \in Y$, if there exists i such that $2 \leq i \leq n-2$ and $yv_i \in E'_1$, then we have a P_n in $T'(E'_1)$, which is a contradiction to

the choice of P' . Thus $[V(P'), Y]_{T'} \subseteq E'_2$. As $\delta(T'(E'_1)) \geq n - 2$, we have $T'[V(P')] \subseteq T'(E'_1)$ and $T'[Y] \subseteq T'(E'_1)$. Note that $T'[V(P')] \cong K_{n-1}$ and $T'[Y] \cong K_{n-1}$. If there exist $v_i \in V(C')$ and $y \in Y$ such that $av_i, ay \in E_1$, then we have a P_{2n-1} in $T(E_1)$, a contradiction. Otherwise, $[a, V(C')]_T \subseteq E_2$ or $[a, Y]_T \subseteq E_2$. If $[a, V(C')]_T \subseteq E_2$, then take any $v_i \neq b$ in $V(C')$; now we have a $K_{1,n}$ in $T(E_2)$ with centre v_i and pendant vertices in $\{a\} \cup Y$. If $[a, Y]_T \subseteq E_2$, then take any $y \neq b$ in Y ; now we have a $K_{1,n}$ in $T(E_2)$ with centre y and pendant vertices in $\{a\} \cup V(C')$.

This completes the proof of the lemma. \square

Theorem 3.6. *If $n \leq m \leq 2n - 1$, then $de(P_m, K_{1,n}) > 1$.*

Proof. As $R(P_m, K_{1,n}) = 2n - 1$, we need to prove that any 2-edge-colouring of $K_{2n-1} - E(K_{1,1})$ contains a P_m in colour 1 or a $K_{1,n}$ in colour 2. This follows from Lemma 3.5 along with the fact that $P_m \subseteq P_{2n-1}$. \square

Theorem 3.7. *If $n + 1 \leq m \leq 2n - 2$, then $de(P_m, K_{1,n}) \leq n - 1$.*

Proof. As $R(P_m, K_{1,n}) = 2n - 1$, we need to provide a 2-edge-colouring (F_1, F_2) to $T = K_{2n-1} - E(K_{1,n-1})$ such that neither $T(F_1)$ has a P_m nor $T(F_2)$ has a $K_{1,n}$. In T , there exists a vertex, say, a , such that $d_T(a) = n - 1$. Let $A = N_T[a]$ and $B = V(T) \setminus A$. Let $T(F_1) = T[A] \cup T[B]$. Colour the remaining edges of T by colour 2. As $T(F_1) \cong K_n \cup K_{n-1}$, it must be that $T(F_1)$ has no P_m , since $m \geq n + 1$. Note that there are no colour 2 edges incident at a . Since $T(F_2) \cong K_{n-1, n-1}$, we have that $T(F_2)$ has no $K_{1,n}$. This completes the proof. \square

Lemma 3.8. *For $n \geq 3$ and $m \geq 2n - 1$, any 2-edge-colouring of $K_m - e$ contains a P_m in colour 1 or a $K_{1,n}$ in colour 2.*

Proof. We use induction on m . For $m = 2n - 1$, this follows from Lemma 3.5. So, assume $m \geq 2n$ and let (E_1, E_2) be a 2-edge-colouring of $T = K_m - e$ where $e = ab$. Consider $T' = T - v$ where $v \notin \{a, b\}$. The 2-edge-colouring (E_1, E_2) of T induces a 2-edge-colouring (E'_1, E'_2) in T' . Observe that $T' \cong K_{m-1} - e$. As $m - 1 \geq 2n - 1$, by induction hypothesis, $T'(E'_1)$ has a P_{m-1} , or $T'(E'_2)$ has a $K_{1,n}$. If $T'(E'_2)$ has a $K_{1,n}$, we are done. Otherwise, $T'(E'_1)$ has a P_{m-1} . Let $P' := v_1v_2 \dots v_{m-1}$ be a P_{m-1} in $T'(E'_1)$. If $vv_1 \in E_1$ or $vv_{m-1} \in E_1$, then we have a P_m in $T(E_1)$. Otherwise, $vv_1, vv_{m-1} \in E_2$.

Similarly, if there exists $i \in \{1, 2, \dots, m-2\}$ such that $v_i v v_{i+1} \subseteq T(E_1)$, then $v_1 v_2 \dots v_i v v_{i+1} v_{i+2} \dots v_{m-2} v_{m-1}$ is a P_m in $T(E_1)$. Otherwise, for every $i \in \{1, 2, \dots, m-2\}$, we have $v_i v \in E_2$ or $v_{i+1} v \in E_2$. These imply that $d_{T(E_2)}(v) \geq \lceil \frac{m}{2} \rceil \geq n$, and so we have a $K_{1,n}$ in $T(E_2)$. \square

Theorem 3.9. *If $n \geq 3$ and $m \geq 2n - 1$, then $de(P_m, K_{1,n}) > 1$.*

Proof. As $R(P_m, K_{1,n}) = m$, it is enough to prove that any 2-edge-colouring of $K_m - E(K_{1,1})$ contains a P_m in colour 1 or a $K_{1,n}$ in colour 2, which follows from the above lemma. \square

From Theorem 1.2, we see that it is enough to consider $m \leq 2n$. By Theorems 3.6 and 3.9, we have the following.

Theorem 3.10. *If $n \geq 3$ and $n \leq m \leq 2n$, then $de(P_m, K_{1,n}) > 1$.*

Theorem 3.11. *If $n \equiv 0$ or $2 \pmod{m-1}$, then $de(P_m, K_{1,n}) \leq m-2$.*

Proof. As $R(P_m, K_{1,n}) = m+n-2$, we need to prove that there exists a 2-edge-colouring (F_1, F_2) to $T = K_{m+n-2} - E(K_{1,m-2})$ such that neither $T(F_1)$ contains a P_m nor $T(F_2)$ contains a $K_{1,n}$. In T , there exists a vertex, say a , such that $d_T(a) = n-1$.

If $n \equiv 0 \pmod{m-1}$, then $n = p(m-1)$ for some $p \geq 1$. This implies that $m+n-2 = p(m-1) + (m-2)$. Let $A = N_T[a]$ and $B = V(T) \setminus A$. As $|A| = n$, we can write $A = \bigcup_{i=1}^p V_i$, where $|V_i| = m-1$ for every $i \in \{1, 2, \dots, p\}$. Let $T(F_1) = \bigcup_{i=1}^p T[V_i] \cup T[B]$. Colour the remaining edges of T by colour 2. As $T(F_1) \cong pK_{m-1} \cup K_{m-2}$, it must be that $T(F_1)$ does not contain P_m . Observe that $d_{T(F_2)}(a) = (p-1)(m-1) = n-m+1 \leq n-1$ and, for every $v \neq a$, we have $d_{T(F_2)}(v) = (p-1)(m-1) + (m-2) = n-1$. So, $T(F_2)$ does not contain a $K_{1,n}$.

If $n \equiv 2 \pmod{m-1}$, then $n-2 = p(m-1)$ for some $p \geq 1$. This implies that $m+n-2 = m+p(m-1)$. Let $A \subseteq N_T(a)$ be such that $|A| = m-1$ and $B = V(T) \setminus (A \cup \{a\})$. As $|B| = n-2$, we can write $B = \bigcup_{i=1}^p V_i$, where $|V_i| = m-1$ for every $i \in \{1, 2, \dots, p\}$. Let $T(F_1) = T[A] \cup \bigcup_{i=1}^p T[V_i]$. Colour the remaining edges of T by colour 2. As $T(F_1) \cong (p+1)K_{m-1}$, it must be that $T(F_1)$ does not contain a P_m . If $v \notin N_T[a]$ (resp. $v \in N_T[a]$), then $d_{T(F_2)}(v) = (p-1)(m-1) + m-1 = p(m-1) = n-2$ (resp. $d_{T(F_2)}(v) = (p-1)(m-1) + m-1 + 1 = p(m-1) + 1 = n-1$). So, $T(F_2)$ does not contain a $K_{1,n}$. \square

Theorem 3.12. *If $m, n \geq 3$ while p and q are positive integers with*

$$R(P_m, K_{1,n}) = m + n - 2 = p(m - 2) + q(m - 1),$$

then $de(P_m, K_{1,n}) \leq p(m - 2)$.

Proof. We need to prove that there exists a 2-edge-colouring (F_1, F_2) to $T = K_{m+n-2} - E(K_{1,p(m-2)})$ such that neither $T(F_1)$ contains a P_m nor $T(F_2)$ contains a $K_{1,n}$. In T , there exists a vertex, say a , such that $d_T(a) = q(m - 1) - 1$. Let $A \subseteq N_T[a]$ be such that $a \in A$ and $|A| = m - 1$ and let $B = V(T) \setminus A$. As $|N_T[a] \setminus A| = (q - 1)(m - 1)$, we can write $(N_T[a] \setminus A) = \bigcup_{j=1}^{q-1} V_j$ such that $|V_j| = m - 1$ for every $j \in \{1, 2, 3, \dots, q-1\}$. As $|V(T) \setminus N_T[a]| = p(m - 2)$, we can write $(V(T) \setminus N_T[a]) = \bigcup_{i=1}^p U_i$ such that $|U_i| = m - 2$ for every $i \in \{1, 2, 3, \dots, p\}$. Let $T(F_1) = T[A] \cup \bigcup_{i=1}^p T[U_i] \cup \bigcup_{j=1}^{q-1} T[V_j]$. Colour the remaining edges of T by colour 2. As $T(F_1) \cong pK_{m-2} \cup qK_{m-1}$, it must be that $T(F_1)$ does not contain a P_m . Observe that $d_{T(F_2)}(a) = (q - 1)(m - 1) = n - 1 - p(m - 2)$ and, for every $v \neq a$, we have $d_{T(F_2)}(v) = p(m - 2) + (q - 1)(m - 1) = n - 1$. So, $T(F_2)$ does not contain a $K_{1,n}$. \square

Remark 3.13. If $n \not\equiv 0, 1, \text{ or } 2 \pmod{m - 1}$, $n \not\equiv 1 \pmod{m - 2}$, and $n \geq (m - 3)^2$, then the hypothesis of Theorem 3.12 is satisfied.

To see this, we need the following lemma from [9].

Lemma 3.14 (Parsons, [9]). *Let a, M be positive integers. If $M \geq a^2 - a$, then there exist non-negative integers p, q such that $M = pa + q(a + 1)$.*

Proof of Remark 3.13. As $n \geq (m - 3)^2$, we also have that $m + n - 2 > m + n - 3 \geq (m - 2)^2 - (m - 2)$. By Lemma 3.14 (take $M = m + n - 2$ and $a = m - 2$), there exist non-negative integers p, q such that $m + n - 2 = p(m - 2) + q(m - 1)$. If $p = 0$ (resp. $q = 0$), then $n \equiv 1 \pmod{m - 1}$ (resp. $n \equiv 1 \pmod{m - 2}$), a contradiction. Thus $p, q \geq 1$. \square

4 $K_{1,3}$ versus $K_{m,n}$

Harborth and Mengersen [4] computed that $R(K_{1,3}, K_{m,n}) = m + n + 2$, $n \geq m \geq 1$. In this section, we give an upper bound to $de(K_{1,3}, K_{m,n})$. For any graph G , we use G^c to denote the complement of G .

Observation 4.1. Let $n \geq m \geq 2$ and $G = C_{m+n+1}$.

- (i) G^c contains $K_{i,j}$ if and only if $i + j \leq m + n - 1$.
- (ii) If (X, Y) is the bipartition of $K_{m-1,n} \subseteq G^c$, then $G[X] \cong P_{m-1}$, $G[Y] \cong P_n$, and $d_G(X, Y) = 2$, where $d_G(X, Y) = \min\{d_G(x, y) : x \in X \text{ and } y \in Y\}$.

Proof. Let $G = C_{m+n+1} := v_0v_1v_2 \dots v_{m+n}v_0$. Clearly, both $G[X]$ and $G[Y]$ are disjoint unions of paths. If $v_i \in X$ (resp. $v_i \in Y$), then, as $v_iv_{i-1}, v_iv_{i+1} \in E(G)$, we have $v_{i-1}, v_{i+1} \notin Y$ (resp. $v_{i-1}, v_{i+1} \notin X$). Hence, $d_G(X, Y) \geq 2$. As $m + n + 1 = |X| + |Y| + 2$, we have $d_G(X, Y) = 2$, $G[X] \cong P_{m-1}$, and $G[Y] \cong P_n$. \square

- (iii) If (U, V) is the bipartition of $K_{m,n-1} \subseteq G^c$, then $G[U] \cong P_m$, $G[V] \cong P_{n-1}$, and $d_G(U, V) = 2$.

Theorem 4.2. If $n \geq m \geq 2$, then $de(K_{1,3}, K_{m,n}) \leq \lceil \frac{n+1}{m} \rceil + 1$.

Proof. As $R(K_{1,3}, K_{m,n}) = m+n+2$, we need to provide a 2-edge-colouring to $K_{m+n+2} - E(K_{1, \lceil \frac{n+1}{m} \rceil + 1})$ such that there is no $K_{1,3}$ in colour 1 and no $K_{m,n}$ in colour 2. First, consider $T' = K_{m+n+1}$.

Let $V(T') = \{v_0, v_1, v_2, \dots, v_{m+n}\}$, where subscripts are reduced modulo $m + n + 1$. Define a 2-edge-colouring (F'_1, F'_2) to T' as follows: $F'_1 = \{v_0v_1, v_1v_2, v_2v_3, \dots, v_{m+n-1}v_{m+n}, v_{m+n}v_0\}$ and $F'_2 = E(T') \setminus F'_1$. Now, define T by adding a new vertex a to T' such that

$$E(T) = E(T') \cup \{av_i : \text{neither } i + 1 \equiv 0 \pmod{m} \text{ nor } i = m + n\}.$$

Colour all the edges incident at a by 2. Denote the 2-edge-colouring of T by (F_1, F_2) . Note that $F_1 = F'_1$ and $F_2 = F'_2 \cup [a, V(T')]_T$. Since

$$|\{v_i : i + 1 \equiv 0 \pmod{m} \text{ or } i = m + n\}| = \lceil \frac{n+1}{m} \rceil + 1,$$

then

$$|[a, V(T')]_T| = |[a, V(T')]_{T(F_2)}| = m + n + 1 - (\lceil \frac{n+1}{m} \rceil + 1).$$

Thus $T \cong K_{m+n+2} - E(K_{1, \lceil \frac{n+1}{m} \rceil + 1})$, where the partite sets of the star $K_{1, \lceil \frac{n+1}{m} \rceil + 1}$ are $\{a\}$ and $\{v_i : i + 1 \equiv 0 \pmod{m} \text{ or } i = m + n\}$. Clearly, $T(F_1)$ is 2-regular, and so $T(F_1)$ does not contain a $K_{1,3}$. Now, it remains to prove that $T(F_2)$ does not contain $K_{m,n}$.

Suppose $T(F_2)$ contains a $K_{m,n}$, say, $H(X, Y)$. As $T'(F'_1) \cong C_{m+n+1}$, by Observation 4.1(i), $T'(F'_2)$ does not contain $K_{m,n}$. So, $a \in V(H)$.

If $a \in X$, then $H \setminus \{a\} \cong K_{m-1,n} \subseteq T'(F'_2)$ is bipartite with bipartition $(X \setminus \{a\}, Y)$. By Observation 4.1(ii), there exists $j \in \{0, 1, \dots, m+n\}$ such that $Y = \{v_j, v_{j+1}, \dots, v_{j+n-1}\}$. As the vertices of Y are n consecutive vertices of C_{m+n+1} and $n \geq m$, we have a $k \in \{j, j+1, \dots, j+n-1\}$ such that $k \equiv -1 \pmod{m}$ or $k = m+n$. This implies that $av_k \in F_2$, a contradiction.

If $a \in Y$, then $H \setminus \{a\} \cong K_{m,n-1}$. Again, by using Observation 4.1(iii), we arrive at a similar kind of contradiction.

This proves that $T(F_2)$ does not contain a $K_{m,n}$. □

5 Conclusion

We summarise our results in terms of $r^*(G, H)$. For $m, n \geq 3$, in Section 3, we showed that

$$\begin{aligned} r^*(P_m, K_{1,n}) &= 1, & \text{if } n \equiv 1 \pmod{m-1}; \\ r^*(P_m, K_{1,n}) &\geq n, & \text{if } n \equiv 0 \text{ or } 2 \pmod{m-1}; \\ r^*(P_n, K_{1,n}) &\leq 2n-3; \\ n \leq r^*(P_m, K_{1,n}) &\leq 2n-3, & \text{if } n+1 \leq m \leq 2n-1; \\ n \leq r^*(P_{2n}, K_{1,n}) &\leq 2n-2. \end{aligned}$$

In Section 4, we showed that, for $n \geq m \geq 2$,

$$r^*(K_{1,3}, K_{m,n}) \geq m + (n+1) - \left\lceil \frac{n+1}{m} \right\rceil.$$

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