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Computational lower bounds for weakened Ramsey numbers from strongly regular t-colorings of complete graphs

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Abstract. We introduce and study strongly regular t-colorings of complete graphs and prove some properties that extend well-known properties of strongly regular graphs. As an example, we consider a coloring that corresponds with a partition of the edges of a complete graph into congruence classes modulo the set of t^{th} powers modulo a given prime number. We then show how a modified version of an algorithm for cubic residue graphs due to Su, Li, Luo, and Li (2002) can be used computationally to find new lower bounds for 26 weakened Ramsey numbers.

1 Context and significance

Throughout this paper all graphs are assumed to be simple (no loops or multi-edges) and undirected. Fields are of prime order (for simplicity of presentation, although finite fields of any order can be substituted with very little work). The t-color Ramsey number for a graph G identifies how large a complete graph must be such that, if its edges are colored using t colors, there exists a monochromatic subgraph that is isomorphic to G. If this concept is generalized to allow for subgraphs whose edges use at most s colors, where $1 \leq s < t$, we obtain the weakened Ramsey number. While traditional graph Ramsey theory has a long history both in analytical and computational explorations (see [12] for an authoritative survey on current Ramsey bounds), the exploration of computational bounds for weakened Ramsey numbers is much more recent with some of the first results appearing in [3].

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In this paper, we add to the recent computational work (see for example [1, 11, 14]) in finding improved lower bounds for weakened Ramsey numbers by modifying an algorithm of Su et al. [13] for use in higher residue difference graphs. Our work results in new lower bounds for 26 such numbers. In Section 2, we review the main definitions and provide a brief overview of the relevant background in graph Ramsey theory. In Section 3, we define strongly regular t-colorings of complete graphs and examine their basic properties. Section 4 explores an important example of such colorings, constructed using the t^{th} residues modulo a prime number p. In Section 5, we build on this example by considering the subgraphs spanned by edges using two given colors, and we run an algorithm using Python 3 to obtain new lower bounds for certain weakened Ramsey numbers. The final section closes with some open problems motivated by our work.

2 Ramsey-theoretic background

For a graph $G = (V, E)$, we denote by $|V(G)|$ its order and $|E(G)|$ its size. The complete graph K_n is the graph with order n and size $\binom{n}{2}$. A *t-coloring* of K_n is a map $f\colon E(K_n) \to \{1, 2, \ldots, t\}$. We do not assume that such a map is surjective. For a given graph G, the *t-color Ramsey number* $r^t(G)$ is the least $p \in \mathbb{N}$ such that every t-coloring of K_p contains a monochromatic subgraph that is isomorphic to G. Early on in the study of Ramsey numbers, Paley graphs (quadratic residue difference graphs) played an important role in the determination of lower bounds for $r^2(K_n)$ (see [8]).

A complete subgraph of a graph G is called a *clique*. The order of a largest clique in G is called the *clique number of G* and is denoted $\omega(G)$. In 2002, Su et al. [13] offered an algorithm for determining the clique numbers of cubic residue graphs and their complements. Their work led to several new lower bounds for off-diagonal Ramsey numbers and weakened Ramsey numbers of the form $r_2^3(K_n)$ (see [3]).

In 1977, Chung et al. [5] introduced the first "weakened" version of the t-color Ramsey number (see also [6, 9, 10]). For a graph G and $1 \leq s < t$, define the *weakened Ramsey number* $r_s^t(G)$ to be the least $p \in \mathbb{N}$ such that every t-coloring of K_p contains a subgraph that is isomorphic to G that uses at most s distinct colors on its edges. From these definitions, it follows that $r^t(G) = r_1^t(G)$ and

$$r_s^t(G) \leq r_{s'}^t(G)$$

whenever $s \geq s'$. When G is a complete graph, the known weakened Ramsey numbers with $s \geq 2$ are

$$r_2^3(K_3) = 5 \text{ (see [6])}, \qquad r_2^3(K_4) = 10 \text{ (see [6])}, \qquad r_2^4(K_3) = 5 \text{ (see [10])},$$
$$r_3^4(K_4) = 10 \text{ (see [9])}, \qquad r_4^5(K_4) = r_4^6(K_4) = 7 \text{ (see [9])},$$
$$r_5^6(K_4) = r_5^7(K_4) = r_5^8(K_4) = r_5^9(K_4) = 5 \text{ (see [9])},$$
$$r_4^7(K_4) = 9 \text{ (see [9])}, \qquad r_4^8(K_4) = 10 \text{ (see [9])}, \qquad r_5^{10}(K_4) = 6 \text{ (see [9])}.$$

3 Strongly regular t-colorings

Let G be a d-regular graph of order n that is neither complete nor empty (i.e., all vertices have degree d, where $1 \leq d < n-1$). Then G is called *strongly regular* if there exist integers $\lambda, \mu \geq 0$ such that every adjacent pair of vertices has λ common neighbors and every nonadjacent pair of vertices has μ common neighbors. It is well-known (see [7, Equation (10.1)]) that such a graph satisfies

$$(n-d-1)\mu = d(d-\lambda-1). \tag{1}$$

It is easily observed that a graph G is strongly regular if and only if its complement $\overline{G}$ is strongly regular. This observation leads us to approach generalizing strongly regular graphs from the perspective of coloring edges in complete graphs.

Given a t-coloring $f\colon E(K_n) \to \{1, 2, \dots, t\}$, let

$$E_i := \{e \in E(K_n) \mid f(e) = i\},$$

where $i \in \{1, 2, \dots, t\}$. For distinct vertices $a, b \in V(K_n)$ and $i \in \{1, 2, \dots, t\}$, let

$$N_i(a) := \{x \in V(K_n) \mid f(ax) = i\},$$

and if $ab \in E_i$, let

$$N_{i,k}(a, b) := \{x \in V(K_n) \mid f(ax) = f(bx) = k\}.$$

For $t \geq 2$, define a t-coloring of K_n to be a *strongly regular t-coloring* if for all $i \in \{1, 2, \dots, t\}$ and $a, b, c, d \in V(K_n)$, $|N_i(a)| = |N_i(b)| \geq 1$ and $|N_{i,k}(a, b)| = |N_{i,k}(c, d)|$, for all $k \in \{1, 2, \dots, t\}$. Since these parameters are independent of the specific vertices being considered, we write $d_i := |N_i(a)|$ and $\lambda_{i,k} := |N_{i,k}(a, b)|$. Note that

$$n = 1 + \sum_{i=1}^{t} d_i.$$

Given a strongly regular 2-coloring of a complete graph, the subgraphs spanned by edges in each color are strongly regular graphs. This property does not necessarily hold when $t > 2$, as we shall see in the next section. The following theorem generalizes Equation (1) to strongly regular t-colorings.

Theorem 3.1. *For $t \geq 2$, let f be a strongly regular t-coloring of K_n with parameters d_i and $\lambda_{i,k}$, where $i, k \in \{1, 2, \dots, t\}$. Then for all $k \in \{1, 2, \dots, t\}$,*

$$\sum_{\substack{i \in \{1,2,\dots,t\} \\ i \neq k}} d_i \lambda_{i,k} = d_k(d_k - \lambda_{k,k} - 1).$$

Proof. Select a vertex a and consider the partition of the other vertices into the sets $N_1(a), N_2(a), \dots, N_t(a)$. The theorem follows from counting the number of edges in color k that join $N_k(a)$ to

$$S_k := \bigcup_{\substack{i \in \{1,2,\dots,t\} \\ i \neq k}} N_i(a)$$

in two different ways.

First, note that each vertex in $N_k(a)$ joins to $\lambda_{k,k}$ other vertices in $N_k(a)$. So, each such vertex joins to $d_k - \lambda_{k,k} - 1$ vertices in S_k with edges in color k. As there are d_k vertices in $N_k(a)$, there are a total of $d_k(d_k - \lambda_{k,k} - 1)$ edges in color k joining $N_k(a)$ to S_k. This gives the right side of the equation in the theorem.

To obtain the left side of the equation, consider a vertex in $N_i(a)$ and count the number of edges in color k that join that vertex to vertices in $N_k(a)$. There are $\lambda_{i,k}$ such edges. So, there are a total of $d_i \lambda_{i,k}$ edges in color k joining $N_i(a)$ to $N_k(a)$. Adding up these edges over $i \in \{1, 2, \dots, t\}$, with $i \neq k$, we obtain the total number of edges joining S_k to $N_k(a)$, resulting in the sum given in the theorem. □

To see how the equation given in Theorem 3.1 is a generalization of Equation (1), let $t = 2$ to obtain

$$d_2 \lambda_{2,1} = d_1(d_1 - \lambda_{1,1} - 1).$$

In this case, $d_2 = (n - 1) - d_1$, resulting in the equation

$$(n - d_1 - 1)\lambda_{2,1} = d_1(d_1 - \lambda_{1,1} - 1),$$

which corresponds with Equation (1) when the strongly regular graph considered is the one spanned by edges in color 1.

A t-colored complete graph is called *t-complementary* if the subgraphs spanned by edges in each color are isomorphic to one another. In the next section, we take advantage of the theory of t^{th} residues in finite fields to construct a coloring of a complete graph of prime order that is a strongly regular t-coloring and is t-complementary.

4 The t^{th} residue graph modulo p

Let $p > 2$ be a prime, $\mathbb{F}_p$ be the finite field of order p (which we identify with $\{0, 1, 2, \ldots, p-1\}$), and $\mathbb{F}_p^\times$ be its group of units. Assume that $p \equiv 1 \pmod{2t}$ for $t \geq 2$. Denote by $\mathbb{F}_p^{\times t}$ the subgroup of t^{th} residues of $\mathbb{F}_p^\times$:

$$\mathbb{F}_p^{\times t} := \{x \in \mathbb{F}_p^\times \mid x = y^t \text{ for some } y \in \mathbb{F}_p^\times\}.$$

Note that $\mathbb{F}_p^\times$ is cyclic and the assumed congruence condition assures that $-1 \in \mathbb{F}_p^{\times t}$ (as it defines the unique involution $I\colon \mathbb{F}_p^\times \to \mathbb{F}_p^\times$ given by $x \mapsto -x$). Fix a generator $g \in \mathbb{F}_p^\times$ and define $\Gamma_t(\mathbb{F}_p)$ to be the t-colored K_p whose vertices are identified with the elements in $\mathbb{F}_p$, and for any two distinct vertices $a, b \in \mathbb{F}_p$, edge $ab \in E_i$ if and only if $a - b \in g^i\mathbb{F}_p^{\times t}$, for $i \in \{1, 2, \ldots, t\}$. In Figure 4.1, $\Gamma_3(\mathbb{F}_7)$ is shown.

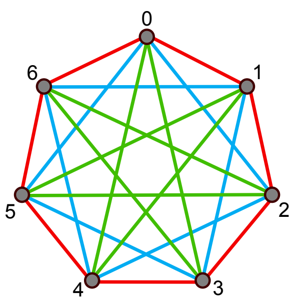

Figure 4.1: The strongly regular 3-coloring $\Gamma_3(\mathbb{F}_7)$.

Let $\Gamma_t^{(i)}(\mathbb{F}_p)$ denote the subgraph of $\Gamma_t(\mathbb{F}_p)$ spanned by the edges in color i. The subgraph $\Gamma_t^{(t)}(\mathbb{F}_p)$ is called the *t^{th} residue graph modulo p* and is a Cayley graph. We show in the next theorem that $\Gamma_t(\mathbb{F}_p)$ is a strongly regular t-coloring, and Figure 4.1 demonstrates that the subgraphs spanned by each color are not necessarily strongly regular graphs when $t > 2$. To see this, note that the endpoints of every blue edge have one common red neighbor, but the endpoints of every green edge have no common red neighbors. As blue and green edges are both in the complement of the subgraph spanned by the red edges, we see that the red subgraph is not strongly regular.

Theorem 4.1. *For any prime $p \equiv 1 \pmod{2t}$, we have $\Gamma_t(\mathbb{F}_p)$ is a strongly regular t-coloring of K_p that is t-complementary.*

Proof. To prove that $\Gamma_t(\mathbb{F}_p)$ is t-complementary, note that the map $x \mapsto g^i x$ defines an isomorphism from $\Gamma_t^{(t)}(\mathbb{F}_p)$ to $\Gamma_t^{(i)}(\mathbb{F}_p)$, for all $i \in \{1, 2, \ldots, t\}$. So, every vertex is incident with exactly $\left|\mathbb{F}_p^{\times t}\right| = \frac{p-1}{t}$ edges in color i. Note that the assumed congruence implies that $\frac{p-1}{t}$ is even, which is required for such a coloring to exist since p is odd.

To prove that $\Gamma_t(\mathbb{F}_p)$ is a strongly regular t-coloring, assume that the edge $ab \in E_i$ and consider the number of subgraphs induced by $\{a, b, c\}$, where $ac, bc \in E_k$. First, we apply the isomorphism that maps vertex x to $x - a$. Note that this map preserves the colors of edges since for any two distinct vertices x and y, $x - y = (x - a) - (y - a)$. Thus, we have that the triangle with vertex set $\{a, b, c\}$ is mapped by this isomorphism to the triangle with vertex set $\{0, (b - a), (c - a)\}$ such that $0(b - a) \in E_i$ and $0(c - b), (b - a)(c - a) \in E_k$.

Next, note that since for any distinct vertices x and y, if $u \in g^t\mathbb{F}_p^{\times t}$ and $x - y \in g^j\mathbb{F}_p^{\times t}$, then $u(x - y) \in g^j\mathbb{F}_p^{\times t}$. So, multiplying by $u \in g^t\mathbb{F}_p^{\times r}$ permutes elements within a coset, never mapping between distinct cosets. There exists a unique $u' \in g^t\mathbb{F}_p^{\times r}$ such that $u'(b - a) = g^i$. Applying the isomorphism $x \mapsto u'x$, the triangle with vertex set $\{0, b - a, c - a\}$ maps to the triangle with vertex set $\{0, g^i, u'(c - a)\}$. Since both of the transformations we have described are isomorphisms, we have that for every edge $ab \in E_i$, the number of vertices c such that $ac, bc \in E_k$ is equal to the number of vertices incident to 0 and g^i via edges in color k. It follows that $\Gamma_t(\mathbb{F}_p)$ is a strongly regular t-coloring of K_p. □

5 Cliques in $\Gamma_t(\mathbb{F}_p)$ spanned by edges using at most s colors

For a prime $p \equiv 1 \pmod{2t}$, we now consider cliques in $\Gamma_t(\mathbb{F}_p)$ spanned by edges using at most s colors, where $1 \leq s < t$. The following theorem will simplify our search for such subgraphs.

Theorem 5.1. *Assume that there exists a clique of order n in $\Gamma_t(\mathbb{F}_p)$ that is spanned by edges using at most s colors, where $1 \leq s < t$. Then there exists a clique of order n spanned by edges using at most s colors that includes the vertices 0 and 1.*

Proof. Suppose that $\{c_1, c_2, \ldots, c_n\}$ is the vertex set for a clique of order $n \geq 2$ that is spanned by edges using at most s colors in $\Gamma_t(\mathbb{F}_p)$. Since the map $x \mapsto x - c_1$ is an automorphism, we see that the subgraph induced by $\{0, c_2 - c_1, \ldots, c_n - c_1\}$ is a clique of order n whose edges use at most s colors. Note that $c_2 - c_1 \in \mathbb{F}_p^\times$, from which it follows that $x \mapsto (c_2 - c_1)^{-1}x$ is an isomorphism. So, the subgraph induced by

$$\{0, 1, (c_2 - c_1)^{-1}c_3, \ldots, (c_2 - c_1)^{-1}c_n\}$$

is a clique of order n whose edges use at most s colors. □

We now focus on the case $s = 2$. For distinct $i, j \in \{1, 2, \ldots, t\}$, let $\Gamma_t^{(i,j)}(\mathbb{F}_p)$ be the subgraph of $\Gamma_t(\mathbb{F}_p)$ spanned by the edges in colors i and j.

Theorem 5.2. *Let $p \equiv 1 \pmod{2t}$ be prime and consider the subgraphs of $\Gamma_t(\mathbb{F}_p)$ spanned by edges using exactly two colors. There are at most $\left\lfloor \frac{t}{2} \right\rfloor$ isomorphism classes for such subgraphs.*

Proof. Without loss of generality, assume that $i < j$ for $i, j \in \{1, 2, \ldots, t\}$. Note that either $1 \leq j - i \leq \left\lfloor \frac{t}{2} \right\rfloor$ or $1 \leq t + i - j \leq \left\lfloor \frac{t}{2} \right\rfloor$. If $1 \leq j - i \leq \left\lfloor \frac{t}{2} \right\rfloor$, then the map $x \mapsto g^{t-i}x$ defines an isomorphism from $\Gamma_t^{(i,j)}(\mathbb{F}_p)$ to $\Gamma_t^{(t,j-i)}(\mathbb{F}_p)$. If $1 \leq t + i - j \leq \left\lfloor \frac{t}{2} \right\rfloor$, then the map $x \mapsto g^{t-j}x$ defines an isomorphism from $\Gamma_t^{(i,j)}(\mathbb{F}_p)$ to $\Gamma_t^{(t+i-j,t)}(\mathbb{F}_p) = \Gamma_t^{(t,t+i-j)}(\mathbb{F}_p)$. In both cases, we find that $\Gamma_t^{(i,j)}(\mathbb{F}_p)$ is isomorphic to $\Gamma_t^{(t,k)}(\mathbb{F}_p)$ for some $k \in \{1, 2, \ldots, \left\lfloor \frac{t}{2} \right\rfloor\}$. □

When determining lower bounds for weakened Ramsey numbers of the form $r_2^t(K_n)$, the previous two theorems imply that we must check $\left\lfloor \frac{t}{2} \right\rfloor$ isomorphism classes of graphs and that we only need to find the maximal cliques that contain both of the vertices 0 and 1 in each case. This leads to a simple algorithm, by finding, for the k^{th} isomorphism class, the set of vertices $N_{t,k}(0) \cap N_{t,k}(1)$, then using a depth-first-search function to find the maximum clique in said set. We present the algorithm now.

In Algorithm 5.1, let H be an empty set, let R, N, and MaxClique be empty lists, and let findClique($N[i]$) be a depth-first search function that looks for the maximum clique in the set $N[i]$. The algorithm returns MaxClique, which is a list of the maximum order of cliques in each of the isomorphism classes.

Algorithm 5.1 Maximum clique size in $\Gamma_t(\mathbb{F}_p)$ spanned by edges using at most 2 colors.

Find a generator of $\mathbb{F}_p^\times$ and call it g.
for $i \in \{1, 2, \ldots, p-1\}$: **do**
    ADD i^t to H
end for
for $i \in \{1, \ldots, \lfloor (t/2) \rfloor\}$: **do**
    $R[i] = H \cup g^i H$
end for
for $i \in \{1, \ldots, \lfloor (t/2) \rfloor\}$: **do**
    for $j \in R[i]$: **do**
        if $j - 1 \in R[i]$: **then**
            ADD j to $N[i]$
        end if
    end for
end for
for $i \in \{1, \ldots, \lfloor (t/2) \rfloor\}$: **do**
    MaxClique[i]=findClique($N[i]$)
end for
return MaxClique

Algorithm 5.1 was implemented in Python 3. The clique numbers obtained are given in Table A.1 in the Appendix. The blue entries in the table imply the lower bounds for the weakened Ramsey numbers given in the following theorem.

Theorem 5.3. *The weakened Ramsey number satisfies the following lower bounds:*

$$\begin{array}{cccc}
r_2^4(K_5) > 17, & r_2^4(K_6) > 89, & r_2^4(K_7) > 97, & r_2^4(K_8) > 137, \\
r_2^4(K_9) > 241, & r_2^4(K_{10}) > 569, & r_2^4(K_{12}) > 881, & r_2^4(K_{14}) > 1009, \\
r_2^5(K_5) > 71, & r_2^5(K_6) > 101, & r_2^5(K_8) > 401, & r_2^5(K_9) > 761, \\
r_2^5(K_{10}) > 1051, & r_2^5(K_{11}) > 1061, & r_2^6(K_4) > 13, & r_2^6(K_5) > 73, \\
r_2^6(K_6) > 277, & r_2^6(K_7) > 733, & r_2^6(K_8) > 1069, & r_2^6(K_9) > 1117, \\
r_2^6(K_{10}) > 1201, & r_2^7(K_5) > 127, & r_2^7(K_7) > 1051, & r_2^7(K_8) > 1877, \\
r_2^4(K_{13}) > 953, & r_2^4(K_{18}) > 1033. & &
\end{array}$$

6 Conclusion

In this paper we have introduced strongly regular t-colorings of complete graphs, we have proven theorems regarding their basic properties, and we have used a modified version of Su's et al. algorithm to find the maximum clique in merged coset graphs of $\Gamma_t(\mathbb{F}_p)$. In a sense, this is the latest step in a long line of computational approaches to Ramsey theory. While this may be an early step in the search for computational bounds for weakened Ramsey numbers, one could, of course, consider other generalizations of classical Ramsey Theory for graphs. Some examples include the following open problems.

- The strongly regular t-coloring $\Gamma_t(\mathbb{F}_q)$ can be considered for any finite field with $q = p^f$ elements, where p is a prime, $f \geq 1$, and $q \equiv 1 \pmod{2t}$. Interestingly, when $t = 3$ and $q = 16$, the graph $\Gamma_3^{(3)}(\mathbb{F}_{16})$ is the Clebsch graph (also called the Greenwood and Gleason graph, as it provided the lower bound for $r^3(K_3)$ in [8]), which is known to be strongly regular. The coloring given in Figure 4.1 provides an example where $t = 3$ and $q = 7$, and the subgraphs spanned by each color are not strongly regular. Is it possible to determine the values of t and q where the subgraphs $\Gamma_t^{(t)}(\mathbb{F}_q)$ are strongly regular?
- Can a similar algorithm be developed for Gallai colorings (i.e., colorings that avoid rainbow K_3-subgraphs)? Such an algorithm might provide new lower bounds for weakened Gallai-Ramsey numbers (see [2, 4]).
- Can a similar construction be developed for weakened hypergraph Ramsey numbers (see [3])?

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Appendix

The following table provides the results of applying Algorithm 5.1 to find the largest cliques in $\Gamma_t(\mathbb{F}_p)$ spanned by edges using at most two colors. The clique numbers given for each prime p correspond with

$$\left[\omega\left(\Gamma_t^{(t,1)}(\mathbb{F}_p)\right), \omega\left(\Gamma_t^{(t,2)}(\mathbb{F}_p)\right), \ldots, \omega\left(\Gamma_t^{(t,\lfloor r/2\rfloor)}(\mathbb{F}_p)\right)\right].$$

The entries given in blue led to the new lower bounds contained in Theorem 5.3.

Table A.1: The results of running Algorithm 5.1.

$t = 4$	$t = 5$	$t = 6$	$t = 7$
p [Cliques]	p [Cliques]	p [Cliques]	p [Cliques]
17 [4 3]	31 [3 4]	13 [3 2 2]	29 [3 2 4]
41 [5 5]	41 [4 4]	37 [3 4 3]	43 [3 3 4]
73 [7 5]	61 [4 4]	61 [5 3 3]	71 [3 4 5]
89 [5 5]	71 [4 4]	73 [4 4 4]	113 [4 4 4]
97 [6 6]	101 [5 4]	97 [4 4 5]	127 [4 4 4]
113 [6 7]	131 [5 7]	109 [4 4 5]	197 [4 4 6]
137 [7 7]	151 [5 7]	157 [4 4 5]	211 [4 5 4]
193 [9 7]	181 [7 7]	181 [7 5 5]	239 [7 6 4]
233 [9 7]	191 [7 6]	193 [6 4 4]	281 [5 7 6]
241 [8 7]	211 [9 5]	229 [7 6 5]	337 [4 4 7]
257 [9 7]	241 [7 9]	241 [8 4 4]	379 [5 4 7]
281 [9 7]	251 [7 5]	277 [5 4 5]	421 [5 4 7]
313 [9 8]	271 [6 9]	313 [7 6 5]	449 [5 5 7]
337 [9 9]	281 [6 8]	337 [5 8 5]	463 [5 7 7]
353 [9 9]	311 [6 6]	349 [5 5 8]	491 [4 5 5]
401 [9 9]	331 [8 7]	373 [6 5 5]	547 [7 4 6]
409 [10 9]	401 [7 7]	397 [7 7 7]	617 [5 7 5]
433 [10 11]	421 [9 6]	409 [6 5 7]	631 [6 6 7]
449 [10 9]	431 [8 8]	421 [6 7 7]	659 [6 5 5]
457 [9 11]	461 [8 7]	433 [7 7 5]	673 [5 9 6]
521 [11 9]	491 [8 8]	457 [8 7 7]	701 [6 5 5]
569 [9 9]	521 [10 7]	541 [7 10 5]	743 [6 6 8]
577 [11 10]	541 [8 7]	577 [7 8 7]	757 [9 5 5]
593 [10 11]	571 [9 7]	601 [6 7 5]	827 [5 8 7]
601 [11 11]	601 [7 9]	613 [7 7 5]	883 [7 7 7]
617 [11 11]	631 [8 7]	661 [7 6 9]	911 [7 6 8]
641 [11 11]	641 [9 9]	673 [7 6 7]	953 [7 6 6]
673 [11 11]	661 [11 7]	709 [7 8 5]	967 [8 7 10]
761 [12 11]	691 [9 8]	733 [6 6 5]	1009 [8 7 7]
769 [13 11]	701 [8 8]	757 [9 7 6]	1051 [5 6 6]
809 [11 11]	751 [11 10]	769 [6 7 8]	1093 [6 8 7]
857 [12 13]	761 [8 8]	829 [7 8 7]	1163 [9 6 7]
881 [11 11]	811 [10 12]	853 [7 10 7]	1289 [6 5 7]
929 [16 13]	821 [9 9]	877 [8 8 5]	1303 [6 7 7]
937 [13 11]	881 [11 8]	937 [6 7 9]	1373 [6 8 9]
953 [12,11]	911 [11 9]	997 [8 9 7]	1429 [7 7 8]
977 [13,13]	941 [8 9]	1009 [8 6 8]	1471 [9 6 8]
1009 [13,11]	971 [8 9]	1021 [8 7 7]	1499 [7 7 7]

Table A.1: (continued)

$t=4$	$t=5$	$t=6$	$t=7$
p [Cliques]	p [Cliques]	p [Cliques]	p [Cliques]
1033 [17,11]	991 [9 9]	1033 [7 11 7]	1583 [7 6 8]
	1021 [9 10]	1069 [7 6 5]	1597 [8 7 13]
	1031 [8 10]	1093 [9 8 7]	1667 [8 8 8]
	1051 [9 8]	1117 [8 8 7]	1709 [7 8 7]
	1061 [8 10]	1201 [9 7 6]	1877 [7 6 6]

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