BULLETIN OF THE ICA
Volume **107** (2026), 26–31

Problem 1

Are there greater expectations for diagonal Ramsey numbers?

Gary Chartrand and Ping Zhang

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H (without isolated vertices), the *Ramsey number* $R(F, H)$ is the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is either a subgraph of K_n isomorphic to F all of whose edges are colored red (a red F) or a subgraph of K_n isomorphic to H all of whose edges are colored blue (a blue H). It is a consequence of a theorem of Ramsey [7] that the number $R(F, H)$ exists for every two graphs F and H. If $F \cong H$, then the (*diagonal*) *Ramsey number* $R(F, H) = R(F, F) = R(F)$ is the minimum positive integer n such that every red-blue coloring of K_n results in a subgraph of K_n isomorphic to F all of whose edges are colored the same (a monochromatic F).

If G is a graph without isolated vertices such that its (diagonal) Ramsey number $R(G) = n$, then for every red-blue coloring of the complete graph K_n of order n, there is a monochromatic subgraph of K_n isomorphic to G. From this fact, the following question arises:

> *For such a graph G, is it possible that every red-blue coloring of K_n has an even stronger property?*

We will return to this question later. Let's now explore some curious properties of all red-blue colorings of the complete graph K_6 of order 6.

Corresponding author: Ping Zhang `<ping.zhang@wmich.edu>`

In 1938 the first William Lowell Putnam Mathematical Competition grew out of a competition between students from Harvard University and cadets from the United States Military Academy. This competition for undergraduate college students from the United States and Canada consists of twelve challenging problems. The 1953 competition contained the following problem.

> **The 1953 Putnam Competition Problem A2**
> *The complete graph with* 6 *points and* 15 *edges has each edge colored red or blue. Show that we can find* 3 *points such that the* 3 *edges joining them are the same color.*

Although this problem does not specifically ask students to show that the Ramsey number $R(K_3, K_3) = R(K_3) = 6$, this is in fact the case as (1) every red-blue coloring of K_6 results in a monochromatic K_3 (a triangle) and (2) there exists a red-blue coloring of K_5 where both the red and blue subgraph is the 5-cycle C_5 and so there is no monochromatic triangle. This coloring is shown in Figure 1, where a bold edge represents a red edge and a thin edge represents a blue edge. In fact, this red-blue coloring of K_5 is the unique such coloring of K_5 that avoids a monochromatic triangle. Thus, the answer to Problem A2 of the 1953 Putnam competition states that the Ramsey number $R(K_3) = 6$. In fact, this result was established by Greenwood and Gleason [4] in 1955.

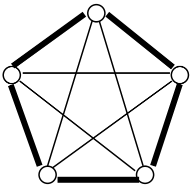

Figure 1: The unique red-blue coloring of K_5 with no monochromatic triangle.

Not only does every red-blue coloring of K_6 produce a monochromatic triangle, it produces at least two monochromatic triangles. For example, the red-blue coloring of K_6 shown in Figure 2 contains exactly two monochromatic (red) triangles. However, for a monochromatic triangle T in this graph, the subgraph $K_6 - E(T)$ contains no monochromatic triangles. That is, it's possible for every two monochromatic triangles in K_6 to have an edge in common.

While the Ramsey number of the 3-cycle $C_3 = K_3$ is 6, the Ramsey number of the 4-cycle C_4 is 6 as well, that is, $R(C_4) = 6$ (see [5, 6], for example).

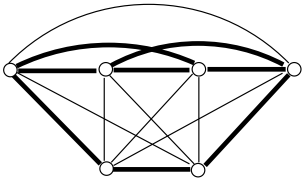

Figure 2: A red-blue coloring of K_6 whose two monochromatic triangles have an edge in common.

Observe that the red-blue coloring of K_5 shown in Figure 1 has no monochromatic C_4. Not only is $R(C_4) = 6$, the Ramsey number of the graph H of Figure 3 obtained by adding a pendant edge to C_4 is also 6. We show this next.

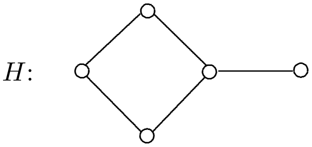

Figure 3: The graph H.

Proposition. $R(H) = 6$.

Proof. Since $C_4 \subseteq H$, it follows that $R(H) \geq R(C_4) = 6$. Hence, it suffices to show that every red-blue coloring of K_6 results in a monochromatic H. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$. Since $R(C_4) = 6$, there is a monochromatic C_4 in G. We may assume that $C_4 = (v_1, v_2, v_3, v_4, v_1)$ is red. If any of the edges v_5v_i, v_6v_i is red for $1 \leq i \leq 4$, then there is a red H. See Figure 4. Thus, we may assume that the all eight edges v_5v_i, v_6v_i $(1 \leq i \leq 4)$ are blue. Thus, $(v_1, v_5, v_2, v_6, v_1)$ is a blue C_4. Since v_5v_3 is blue, there is a blue H in G. Therefore, $R(H) \leq 6$ and so $R(H) = 6$. □

We have mentioned that every red-blue coloring of K_6 produces a monochromatic copy of each of C_3, C_4, and the graph H of Figure 3. However, not only does every red-blue coloring of K_6 result in a monochromatic copy of H, it also produces a sequence H_1, H_2, H_3, H_4, H_5 of five pairwise edge-disjoint monochromatic subgraphs of H such that each subgraph H_i $(1 \leq i \leq 5)$ has size i, where the subgraph H_i is isomorphic to a subgraph of H_{i+1} for $1 \leq i \leq 4$, and $H_5 = H$. For example, consider the red-blue

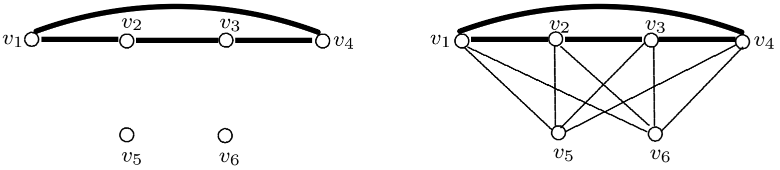

Figure 4: Illustrating the proof of the Proposition.

coloring of K_6 shown in Figure 5, where an edge labeled i ($1 \le i \le 5$) belongs to H_i. Here again, a bold edge represents a red edge and a thin edge represents a blue edge. Then $H_5 \cong H$ is the blue subgraph obtained from $C_4 = (v_2, v_3, v_4, v_5, v_2)$ by adding the pendant edge v_1v_2, the graph H_4 is the blue double star (a tree of diameter 3) whose central vertices have degrees 2 and 3 with $E(H_4) = \{v_1v_3, v_1v_4, v_1v_5, v_4v_2\}$, H_3 is the red star $K_{1,3}$ with $E(H_3) = \{v_6v_1, v_6v_2, v_6v_3\}$, H_2 is the blue 3-path $P_3 = (v_3, v_5, v_6)$, and H_1 is the red 2-path $P_2 = (v_4, v_6)$. In [3] such a sequence is called a *Ramsey chain in K_6 with target graph H*. The concept of a Ramsey chain, introduced by Chartrand [2] in 2021, was inspired by the Ascending Subgraph Decomposition Conjecture, which first appeared in 1987 [1]. The curious feature here is that *every* red-blue coloring of K_6 results in a Ramsey chain with target graph H. Furthermore, the five monochromatic subgraphs in this chain also form a decomposition of K_6.

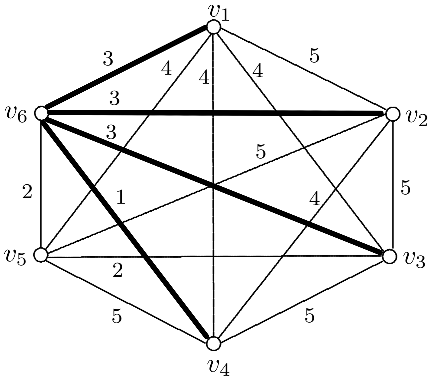

Figure 5: A red-blue coloring of K_6.

In 2025, the *target Ramsey index $TR(G)$* of a graph G of size m with no isolated vertices was introduced in [3] and defined as the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there exists a Ramsey chain $G_1, G_2, \ldots, G_m \cong G$ of m pairwise edge-disjoint monochromatic subgraphs of G such that each subgraph G_i $(1 \leq i \leq m)$ has size i and G_i is isomorphic to a subgraph of G_{i+1} for $1 \leq i \leq m-1$. Such a Ramsey chain is said to have target graph G. Since the target graph in every Ramsey chain must be monochromatic, it follows that $TR(G) \geq R(G)$ for every graph G without isolated vertices. It was shown in [3] that $TR(G)$ exists for every such graph G. For every graph G for which $TR(G)$ has been determined, it has been shown that $TR(G) = R(G)$, that is, *every* red-blue coloring of the complete graph $K_{R(G)}$ of order $R(G)$ not only produces a monochromatic copy of G, but also a Ramsey chain with target graph G.

Let G be graph of size m without isolated vertices such that $TR(G) = n$. Thus, in every red-blue coloring of K_n, there exists a Ramsey chain G_1, G_2, $\ldots$, G_m in K_n where $G_m = G$. Consequently, $|E(K_n)| \geq \sum_{i=1}^{m} i = \binom{m+1}{2}$ and so $\binom{n}{2} \geq \binom{m+1}{2}$. Thus, $TR(G) \geq m+1$. If $TR(G) = R(G) = m+1$, then in every red-blue coloring of $K_{R(G)}$ it follows that every Ramsey chain G_1, G_2, $\ldots$, $G_m \cong G$ in K_n produces a decomposition $\{G_1, G_2, \ldots, G_m\}$ of $K_{R(G)}$. Since $TR(K_2) = R(K_2) = 2$ and $TR(P_3) = R(P_3) = 3$, this is the case for the connected graphs of sizes 1 and 2. In addition, $TR(H) = R(H) = 6$ for the graph H of Figure 3. Since the size of H is 5, this means that for every red-blue coloring of K_6, there exists a decomposition $\{H_1, H_2, H_3, H_4, H_5\}$ of K_6 into five subgraphs H_i $(1 \leq i \leq 5)$ such that (1) H_i has size i, (2) H_i is monochromatic, (3) H_i is isomorphic to a subgraph of H_{i+1} for $1 \leq i \leq 4$, and (4) $H_5 = H$.

These remarks suggest the following question:

> *Is there a graph G such that $TR(G) > R(G)$?*

If the answer to this question is no, then this means that the diagonal Ramsey number $R(G)$ of a graph G possesses a property in addition to the standard properties possessed by diagonal Ramsey numbers. Consequently there are greater expectations for diagonal Ramsey numbers.

References

[1] Y. Alavi, A. J. Boals, G. Chartrand, P. Erdős, and O. R. Oellermann, The ascending subgraph decomposition problem, *Congr. Numer.* **58** (1987), 7–14.

[2] A. Ali, G. Chartrand, and P. Zhang, *Irregularity in Graphs*, Springer, New York (2021).

[3] G. Chartrand and P. Zhang, Target graphs and Ramsey numbers, *Int. J. Comput. Math. Comput. Syst. Theory.*, To appear.

[4] R. E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955), 1–7.

[5] R. J. Faudree and R. H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Math.* **8** (1974), 313–329.

[6] S. P. Radzisowski, Small Ramsey numbers, *Electronic J. Combin.* Dynamic Surveys #**DS1**: Sep 6, 2024, `https://doi.org/10.37236/21`.

[7] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930), 264–286.

G. Chartrand and P. Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, Michigan 49008-5248, USA
gary.chartrand@wmich.edu, ping.zhang@wmich.edu