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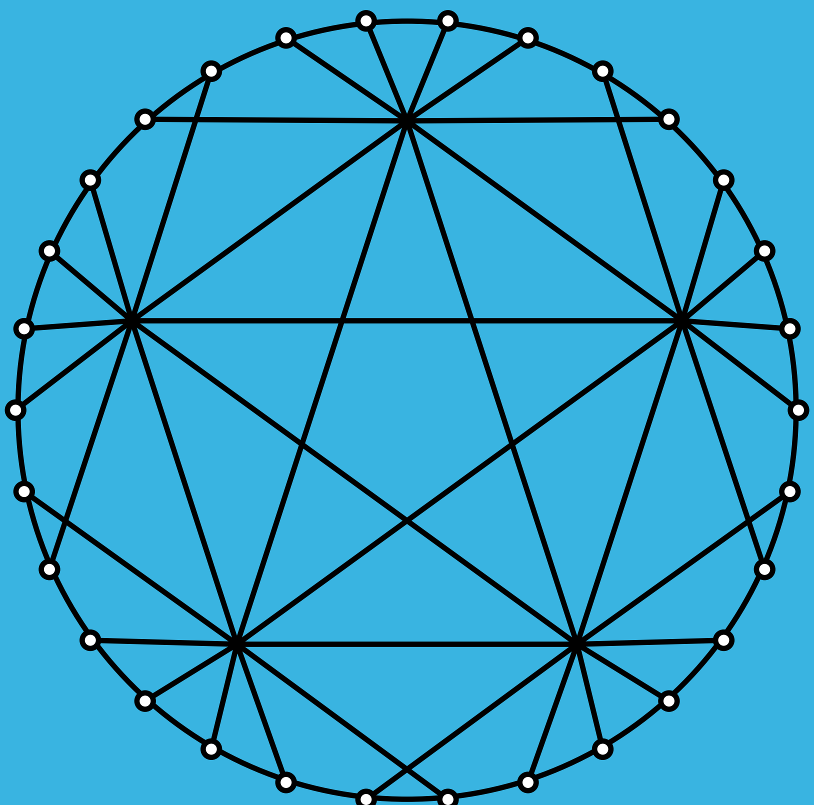

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A note on a definite integral of combinatorial interest

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Abstract. In this note, we consider the numerical sequence $\{I_n^{(s)}\}_{n\geq 0}$, where $I_n^{(s)}$ is the integral of the n-th power of the trinomial $1 + sx + x^2$ on the interval $[0, 1]$. The integrals $I_n^{(s)}$ can be easily expressed in terms of the generalized trinomial coefficients. We show that they can also be expressed in several other different, non-trivial, ways as a single binomial sum. Moreover, we show that the integrals $I_n^{(s)}$, when positive, form a log-convex sequence. Finally, we define a new class of polynomials $I_n^{(s)}(x)$ as a natural extension of the integrals $I_n^{(s)}$ and we show that, for every fixed s, they form an Appell sequence.

1 Introduction

In this note, we will consider the definite integrals

$$I_n^{(s)} = \int_0^1 (1 + sx + x^2)^n \,\mathrm{d}x \qquad (s \in \mathbb{R}) \tag{1}$$

which seems to be essentially[1] absent in the various tables (such as [6, 13] or [5, 2, 12]), even though they are of some interest from a combinatorial

[1] For $s = 0$, or $s = \pm 2$, the trinomial form reduces to a binomial form and in these cases the integrals are known [17]. See also Section 5.

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point of view. These integrals are polynomials in s with rational coefficients, as can be seen from the following first values

$$\begin{aligned}
I_0^{(s)} &= 1,\\
I_1^{(s)} &= \frac{1}{2}\,s + \frac{4}{3},\\
I_2^{(s)} &= \frac{1}{3}\,s^2 + \frac{3}{2}\,s + \frac{28}{15},\\
I_3^{(s)} &= \frac{1}{4}\,s^3 + \frac{8}{5}\,s^2 + \frac{7}{2}\,s + \frac{96}{35},\\
I_4^{(s)} &= \frac{1}{5}\,s^4 + \frac{5}{3}\,s^3 + \frac{184}{35}\,s^2 + \frac{15}{2}\,s + \frac{1328}{315},\\
I_5^{(s)} &= \frac{1}{6}\,s^5 + \frac{12}{7}\,s^4 + \frac{85}{12}\,s^3 + \frac{928}{63}\,s^2 + \frac{31}{2}\,s + \frac{4672}{693},\\
I_6^{(s)} &= \frac{1}{7}\,s^6 + \frac{7}{4}\,s^5 + \frac{188}{21}\,s^4 + \frac{49}{2}\,s^3 + \frac{8752}{231}\,s^2 + \frac{63}{2}\,s + \frac{33472}{3003}.
\end{aligned}$$

Indeed, expanding the power of the trinomial form, we have

$$\begin{aligned}
I_n^{(s)} &= \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} s^k x^k (1+x^2)^{n-k} \right) \mathrm{d}x\\
&= \sum_{k=0}^{n} \binom{n}{k} s^k \int_0^1 x^k (1+x^2)^{n-k}\,\mathrm{d}x\\
&= \sum_{k=0}^{n} \binom{n}{k} s^k \int_0^1 x^k \left(\sum_{i=0}^{n-k} \binom{n-k}{i} x^{2i} \right) \mathrm{d}x\\
&= \sum_{k=0}^{n} \binom{n}{k} s^k \sum_{i=0}^{n-k} \binom{n-k}{i} \int_0^1 x^{2i+k}\,\mathrm{d}x
\end{aligned}$$

that is

$$I_n^{(s)} = \sum_{k=0}^{2n} \binom{n}{k} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} \frac{1}{2i+k+1} \right) s^k\,. \tag{2}$$

Equivalently, we can consider the expansion

$$(1+sx+x^2)^n = \sum_{k=0}^{n} T_{n,k}^{(s)}\,x^k$$

where the coefficients $T_{n,k}^{(s)}$ are the *generalized trinomial coefficients*, which satisfy the recurrence

$$T_{n+1,k+2}^{(s)} = T_{n,k}^{(s)} + s\,T_{n,k+1}^{(s)} + T_{n,k+2}^{(s)}$$

with the initial conditions $T^{(s)}_{n,0} = 1$ and $T^{(s)}_{0,k} = \delta_{k,0}$. We report the first few values in the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & s & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2s & s^2+2 & 2s & 1 & 0 & 0 & 0 & 0 \\ 1 & 3s & 3s^2+3 & s^3+6s & 3s^2+3 & 3s & 1 & 0 & 0 \\ 1 & 4s & 6s^2+4 & 4s^3+12s & s^4+12s^2+6 & 4s^3+12s & 6s^2+4 & 4s & 1 \end{bmatrix}.$$

These coefficients can also be expressed in terms of the binomial coefficients, namely

$$T^{(s)}_{n,k} = \sum_{i=0}^{k} \binom{n}{i} \binom{n-i}{k-2i} s^{k-2i}$$

or

$$T^{(s)}_{n,k} = \sum_{i=0}^{k} \binom{n}{i} \binom{2n-2i}{k-i} (-1)^{k-i} (s+2)^i \,,$$

or, equivalently, in terms of the Gegenbauer polynomials as

$$T^{(s)}_{n,k} = C^{(-n)}_{k}(-s/2)\,.$$

For $s = 1$ we have the ordinary trinomial coefficients $T^{(1)}_{n,k} = \binom{n;3}{k}$, [3, p. 78], which can be interpreted as the number of multisets $\mu : X \to \mathbb{N}$ of order k on a set X of size n, where each element has multiplicity at most 2, i.e. $\mu(x) \in \{0,1,2\}$ for every $x \in X$. The generalized coefficients $T^{(s)}_{n,k}$ can be thought of as the total weight of the set of these multisets where the elements of multiplicity 1 have weight s. For instance, for $n = k = 4$, we have one multiset $\{1,2,3,4\}$ with four different elements (with weight s^4), 12 multisets of the form $\{x,y,z,z\}$ with only one element of multiplicity 2 (each with weight s^2), and 6 multisets of the form $\{x,x,y,y\}$ with $x \neq y$ (each with weight 1). So, the total weight is $T^{(s)}_{4,4} = s^4 + 12s^2 + 6$.

Notice that we also have the relations

$$T^{(-s)}_{n,k} = (-1)^k T^{(s)}_{n,k} \quad (s \in \mathbb{R}),$$

and

$$T^{(-1)}_{n,k} = (-1)^k \binom{n;3}{k}.$$

Clearly, the integrals $I_n^{(s)}$ can also be expressed in terms of the generalized trinomial coefficients. Indeed, we have

$$I_n^{(s)} = \int_0^1 \left(\sum_{k=0}^{2n} T_{n,k}^{(s)} x^k \right) \mathrm{d}x = \sum_{k=0}^{2n} T_{n,k}^{(s)} \int_0^1 x^k \, \mathrm{d}x$$

that is

$$I_n^{(s)} = \sum_{k=0}^{2n} T_{n,k}^{(s)} \frac{1}{k+1} \,. \tag{3}$$

Despite formulas (2) and (3) are very simple to find, the integrals $I_n^{(s)}$ admits other non-trivial (and elegant) representations in terms of a single sum involving only the binomial coefficients. The aim of this note is to derive these representations. In Section 2, we derive two new formulas by means of elementary algebraic manipulations. In Section 3, we find the ordinary generating series of the integrals $I_n^{(s)}$ and then, expanding this series in a suitable way, we obtain a further formula. In Section 4, we show that the integrals $I_n^{(s)}$ satisfy a linear recurrence of the first order and then, solving such a recurrence, we obtain another formula for $I_n^{(s)}$. In Section 5, we consider some special cases where we have some simplifications. In Section 6, we study the log-behavior of the sequence $\{I_n^{(s)}\}_{n\geq 0}$, proving that it is a log-convex sequence, when the integrals are all positive. Finally, in Section 7, we define a new class of polynomials $I_n^{(s)}(x)$, as a natural extension of the integrals $I_n^{(s)}$, and we show that, for a given s, they form an Appell sequence. Furthermore, we show that almost all the formulas and properties obtained for integrals $I_n^{(s)}$ can be extended to the polynomials $I_n^{(s)}(x)$.

2 First results

We can obtain a first formula for the integrals $I_n^{(s)}$ by a simple algebraic manipulation.

Theorem 1. *The integrals $I_n^{(s)}$ can be expressed as*

$$I_n^{(s)} = \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2k+1} \left((s+2)^{2k+1} - s^{2k+1} \right) (4 - s^2)^{n-k} \,. \tag{4}$$

Proof. We have

$$\begin{aligned}
I_n^{(s)} &= \int_0^1 \left(\frac{4-s^2}{4} + \left(x+\frac{s}{2}\right)^2\right)^n \mathrm{d}x \\
&= \int_0^1 \sum_{k=0}^n \binom{n}{k} \left(\frac{4-s^2}{4}\right)^{n-k} (x+s/2)^{2k} \mathrm{d}x \\
&= \sum_{k=0}^n \binom{n}{k} \left(\frac{4-s^2}{4}\right)^{n-k} \int_0^1 (x+s/2)^{2k} \mathrm{d}x \\
&= \sum_{k=0}^n \binom{n}{k} \left(\frac{4-s^2}{4}\right)^{n-k} \left[\frac{(x+s/2)^{2k+1}}{2k+1}\right]_0^1 \\
&= \sum_{k=0}^n \binom{n}{k} \left(\frac{4-s^2}{4}\right)^{n-k} \frac{(1+s/2)^{2k+1}-(s/2)^{2k+1}}{2k+1}
\end{aligned}$$

and this expression simplifies at once in formula (4). □

We can obtain a second formula in a similar way by using the Euler beta function [2, p. 192], defined by

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x \tag{5}$$

for $p, q \in \mathbb{C}$, $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$. This integral can be expressed in terms of the Euler gamma function as $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. In particular, when $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$. Consequently, when $m, n \in \mathbb{N}$, we have

$$B(m+1, n+1) = \frac{1}{\binom{m+n}{m}} \frac{1}{m+n+1}\,. \tag{6}$$

Theorem 2. *The integrals $I_n^{(s)}$ can be expressed as*

$$I_n^{(s)} = \sum_{k=0}^n \binom{n}{k} \frac{(s+2)^{n-k}}{\binom{n+k}{2k}} \frac{1}{n+k+1}\,. \tag{7}$$

Proof. We have

$$I_n^{(s)} = \int_0^1 ((1-x)^2+(s+2)x)^n\,\mathrm{d}x = \sum_{k=0}^n \binom{n}{k} (s+2)^{n-k} \int_0^1 x^{n-k}(1-x)^{2k}\mathrm{d}x\,.$$

By formula (5), we have

$$I_n^{(s)} = \sum_{k=0}^{n} \binom{n}{k} (s+2)^{n-k} B(n-k+1, 2k+1)$$

and by formula (6) we obtain identity (7). $\square$

3 Generating series

Consider the (formal) generating series

$$I^{(s)}(t) = \sum_{n \geq 0} I_n^{(s)} \, t^n \, .$$

By definition (1), we have

$$I^{(s)}(t) = \int_0^1 \left(\sum_{n \geq 0} (1 + sx + x^2)^n \, t^n \right) \mathrm{d}x = \int_0^1 \frac{1}{1 - (1 + sx + x^2)t} \, \mathrm{d}x \, .$$

We will evaluate this integral in a formal way, considering t as an indeterminate (or, possibly, as a negative and very small real number). We have

$$\begin{aligned} I^{(s)}(t) &= \frac{4}{4 + (s^2 - 4)t} \int_0^1 \frac{\mathrm{d}x}{1 + \left(\frac{(2x+s)\sqrt{-t}}{\sqrt{4+(s^2-4)t}} \right)^2} \\ &= \frac{2}{\sqrt{-t}\sqrt{4 + (s^2 - 4)t}} \left[\arctan \frac{(2x + s)\sqrt{-t}}{\sqrt{4 + (s^2 - 4)t}} \right]_0^1 \end{aligned}$$

that is

$$I^{(s)}(t) = \frac{2}{\sqrt{-t}\sqrt{4+(s^2-4)t}} \left(\arctan \frac{(s+2)\sqrt{-t}}{\sqrt{4+(s^2-4)t}} - \arctan \frac{s\sqrt{-t}}{\sqrt{4+(s^2-4)t}} \right) . \tag{8}$$

If we consider the usual expansion of the arc tangent series

$$\arctan z = \sum_{k \geq 0} \frac{(-1)^k}{2k + 1} \, z^{2k+1} \, , \tag{9}$$

then it is straightforward to reobtain formula (4). However, if we consider Euler's expansion

$$\arctan z = \sum_{k\geq 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{2k+1} \frac{z^{2k+1}}{(1+z^2)^{k+1}}\,, \tag{10}$$

then we obtain a new formula, as proved in the following theorem.

Theorem 3. *The integrals $I_n^{(s)}$ can be expressed as*

$$I_n^{(s)} = \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\binom{2k}{k}} \frac{1}{2k+1} \left((s+2)^{n+k+1} - s^{2k+1}\right). \tag{11}$$

Proof. From series (8) and expansion (10), we have

$$\begin{aligned}
&I^{(s)}(t)\\
&= \frac{2}{\sqrt{-t}\sqrt{4+(s^2-4)t}} \left(\sum_{k\geq 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{2k+1} \frac{(s+2)^{2k+1}}{\left(1-\frac{(s+2)^2 t}{4+(s^2-4)t}\right)^{k+1}} \left(\frac{-t}{4+(s^2-4)t}\right)^{k+1/2} \right.\\
&\qquad \left. - \sum_{k\geq 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{2k+1} \frac{s^{2k+1}}{\left(1-\frac{s^2 t}{4+(s^2-4)t}\right)^{k+1}} \left(\frac{-t}{4+(s^2-4)t}\right)^{k+1/2} \right)\\
&= \frac{2}{4+(s^2-4)t} \left(\sum_{k\geq 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{(s+2)^{2k+1}}{2k+1} \left(\frac{4+(s^2-4)t}{4-4(s+2)t}\right)^{k+1} \left(\frac{-t}{4+(s^2-4)t}\right)^{k} \right.\\
&\qquad \left. - \sum_{k\geq 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{s^{2k+1}}{2k+1} \left(\frac{4+(s^2-4)t}{4-4t}\right)^{k+1} \left(\frac{-t}{4+(s^2-4)t}\right)^{k} \right)\\
&= 2\left(\sum_{k\geq 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{(s+2)^{2k+1}}{2k+1} \frac{1}{4^{k+1}} \frac{(-t)^k}{(1-(s+2)t)^{k+1}} - \sum_{k\geq 0} \frac{2^{2k}}{\binom{2k}{k}} \frac{s^{2k+1}}{2k+1} \frac{1}{4^{k+1}} \frac{(-t)^k}{(1-t)^{k+1}} \right)\\
&= \frac{1}{2}\sum_{k\geq 0} \frac{(-1)^k}{\binom{2k}{k}} \frac{1}{2k+1} \left((s+2)^{k+1} \frac{((s+2)t)^k}{(1-(s+2)t)^{k+1}} - s^{2k+1} \frac{t^k}{(1-t)^{k+1}} \right)\\
&= \frac{1}{2}\sum_{k\geq 0} \frac{(-1)^k}{\binom{2k}{k}} \frac{1}{2k+1} \left((s+2)^{k+1} \sum_{n\geq 0} \binom{n}{k} (s+2)^n t^n - s^{2k+1} \sum_{n\geq 0} \binom{n}{k} t^n \right)\\
&= \frac{1}{2}\sum_{n\geq 0} \left(\sum_{k\geq 0} \binom{n}{k} \frac{(-1)^k}{\binom{2k}{k}} \frac{1}{2k+1} \left((s+2)^{n+k+1} - s^{2k+1}\right) \right) t^n\,.
\end{aligned}$$

Taking the coefficient of t^n in the first and last series, we obtain formula (11). $\square$

It is also possible to obtain a more complex representation of the integrals $I_n^{(s)}$ in terms of a double sum, as proved in the next theorem.

Theorem 4. *The integrals $I_n^{(s)}$ can be expressed as*

$$I_n^{(s)} = \sum_{k=0}^{n} \frac{(s+2)^{n-k}}{2k+1} \sum_{i=0}^{k} \binom{k}{i} \binom{n+i}{2k} \frac{(s-2)^{k-i}}{2^{n-k}} \,. \tag{12}$$

Proof. Using the addition formula for the arc tangent, series (8) becomes

$$I^{(s)}(t) = \frac{2}{\sqrt{-t}\sqrt{4+(s^2-4)t}} \arctan \frac{\sqrt{-t}\sqrt{4+(s^2-4)t}}{2-(s+2)t}. \tag{13}$$

Then, using expansion (9), it is straightforward to obtain identity (12). □

4 Recurrence

To derive a recurrence for the integrals $I_n^{(s)}$, we will consider the following further integrals

$$J_n^{(s)} = \int_0^1 x\,(1+sx+x^2)^n \,\mathrm{d}x\,, \qquad H_n^{(s)} = \int_0^1 x^2(1+sx+x^2)^n \,\mathrm{d}x\,.$$

From definition (1), we have at once the identity

$$I_{n+1}^{(s)} = I_n^{(s)} + s\,J_n^{(s)} + H_n^{(s)}\,. \tag{14}$$

On the other hand, integrating by parts, we have

$$I_{n+1}^{(s)} = \Big[x\,(1+sx+x^2)^{n+1}\Big]_0^1 - (n+1)\int_0^1 (sx+2x^2)\,(1+sx+x^2)^n \,\mathrm{d}x$$

that is

$$I_{n+1}^{(s)} = (s+2)^{n+1} - (n+1)s\,J_n^{(s)} - 2(n+1)H_n^{(s)}\,. \tag{15}$$

Finally, we have

$$s\,I_n^{(s)} + 2\,J_n^{(s)} = \int_0^1 (s+2x)\,(1+sx+x^2)^n \,\mathrm{d}x = \left[\frac{(1+sx+x^2)^{n+1}}{n+1}\right]_0^1$$

that is

$$s\,I_n^{(s)} + 2\,J_n^{(s)} = \frac{(s+2)^{n+1}-1}{n+1}\,.$$

From this last equation, we obtain

$$J_n^{(s)} = \frac{1}{2}\left(\frac{(s+2)^{n+1}-1}{n+1} - s\, I_n^{(s)}\right). \tag{16}$$

By equation (15), we have

$$H_n^{(s)} = \frac{1}{2(n+1)}\left((s+2)^{n+1} - (n+1)s\, J_n^{(s)} - I_{n+1}^{(s)}\right)$$

and then, by equation (16), we obtain

$$H_n^{(s)} = \frac{(2-s)(s+2)^{n+1}+s}{4(n+1)} + \frac{s^2}{4}\, I_n^{(s)} - \frac{1}{2(n+1)}\, I_{n+1}^{(s)}. \tag{17}$$

Finally, by replacing (16) and (17) in equation (14) and by simplifying, we obtain the following theorem.

Theorem 5. *The integrals $I_n^{(s)}$ are defined by the recurrence*

$$I_{n+1}^{(s)} = \frac{4-s^2}{2}\,\frac{n+1}{2n+3}\, I_n^{(s)} + \frac{(s+2)^{n+2}-s}{2(2n+3)} \tag{18}$$

with the initial value $I_0^{(s)} = 1$.

Notice that (18) is a linear recurrence of the first order of the form

$$y_{n+1} = a_{n+1}y_n + b_{n+1}\,.$$

If $a_n \neq 0$ for every $n \geq 1$, we have the solution

$$y_n = a_n^* y_0 + \sum_{k=1}^{n} \frac{a_n^*}{a_k^*}\, b_k$$

where $a_n^* = a_n a_{n-1} \cdots a_2 a_1$. Moreover, if $y_0 = b_0$, then the solution can be written as

$$y_n = \sum_{k=0}^{n} \frac{a_n^*}{a_k^*}\, b_k\,. \tag{19}$$

This simple remark implies the following theorem.

Theorem 6. *The integrals $I_n^{(s)}$ can be expressed as*

$$I_n^{(s)} = \frac{1}{2}\,\frac{1}{\binom{2n}{n}}\,\frac{1}{2n+1}\sum_{k=0}^{n}\binom{2k}{k}\left((s+2)^{k+1}-s\right)(4-s^2)^{n-k}\,. \tag{20}$$

Proof. From recurrence (18), we have the coefficients

$$a_n = \frac{4-s^2}{2}\,\frac{n}{2n+1}\,, \qquad b_n = \frac{(s+2)^{n+1}-s}{2(2n+1)}\,. \tag{21}$$

Hence, we have

$$\begin{aligned} a_n^* &= \left(\frac{4-s^2}{2}\right)^n \frac{n(n-1)\cdots 1}{(2n+1)(2n-1)\cdots 3} \\ &= \left(\frac{4-s^2}{2}\right)^n \frac{2^n\, n!\, n!}{(2n+1)!} \\ &= \frac{(4-s^2)^n}{\binom{2n}{n}}\,\frac{1}{2n+1}\,. \end{aligned}$$

Replacing in solution (19), we obtain at once formula (20). □

5 Special cases

As an exemplification of what we have obtained in the previous sections, we now consider the special cases $s = \pm 1$, $s = \pm 2$ and $s = 0$, where we have some small simplifications.

For $s = \pm 1$, the first few values of our integrals (for $0 \leq n \leq 10$) are:

$$1,\ \tfrac{11}{6},\ \tfrac{37}{10},\ \tfrac{1133}{140},\ \tfrac{11869}{630},\ \tfrac{42445}{924},\ \tfrac{463979}{4004},\ \tfrac{5144257}{17160},\ \tfrac{19216887}{24310},\ \tfrac{1954312829}{923780},\ \tfrac{22227331675}{3879876}$$

and

$$1,\ \tfrac{5}{6},\ \tfrac{7}{10},\ \tfrac{83}{140},\ \tfrac{319}{630},\ \tfrac{403}{924},\ \tfrac{1517}{4004},\ \tfrac{1139}{3432},\ \tfrac{1425}{4862},\ \tfrac{48199}{184756},\ \tfrac{907741}{3879876}.$$

In these cases, identities (3), (4), (7), (11) and (20) become

$$\begin{aligned} I_n^{(1)} &= \sum_{k=0}^{2n} \binom{n;3}{k} \frac{1}{k+1}, \\ I_n^{(1)} &= \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(3^{2k+1}-1)\,3^{n-k}}{2k+1}, \\ I_n^{(1)} &= \sum_{k=0}^{n} \binom{n}{k} \frac{3^{n-k}}{\binom{n+k}{2k}} \frac{1}{n+k+1}, \end{aligned}$$

$$I_n^{(1)} = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{\binom{2k}{k}}\frac{3^{n+k+1}-1}{2k+1}$$

$$I_n^{(1)} = \frac{1}{2}\frac{1}{\binom{2n}{n}}\frac{1}{2n+1}\sum_{k=0}^{n}\binom{2k}{k}(3^{k+1}-1)\,3^{n-k}.$$

and

$$I_n^{(-1)} = \sum_{k=0}^{2n}\binom{n;3}{k}\frac{(-1)^k}{k+1},$$

$$I_n^{(-1)} = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{n}{k}\frac{3^{n-k}}{2k+1},$$

$$I_n^{(-1)} = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{\binom{n+k}{2k}}\frac{1}{n+k+1},$$

$$I_n^{(-1)} = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{\binom{2k}{k}}\frac{1}{2k+1},$$

$$I_n^{(-1)} = \frac{1}{\binom{2n}{n}}\frac{1}{2n+1}\sum_{k=0}^{n}\binom{2k}{k}3^{n-k}.$$

For $s = \pm 2$, the integrals can be calculated explicitly, since

$$I_n^{(2)} = \int_0^1 (1+x)^{2n}\,\mathrm{d}x = \frac{2^{2n+1}-1}{2n+1}\,,$$

$$I_n^{(-2)} = \int_0^1 (1-x)^{2n}\,\mathrm{d}x = \frac{1}{2n+1}\,.$$

For $s = 2$, formulas (3) and (4) lead to trivial identities, while formulas (7) and (11) lead to the identities

$$\sum_{k=0}^{n}\binom{n}{k}\frac{4^{n-k}}{\binom{n+k}{2k}}\frac{1}{n+k+1} = \frac{2^{2n+1}-1}{2n+1}\,,$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{\binom{2k}{k}}\frac{4^k}{2k+1} = \frac{1}{2n+1}\,.$$

For $s = -2$, the only non-trivial identity derives from formula (11) and coincides with the second identity obtained in the previous case.

Finally, for $s = 0$, identities (4), (7), (11) and (20) become

$$I_n^{(0)} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2k+1}, \qquad I_n^{(0)} = \sum_{k=0}^{n} \binom{n}{k} \frac{2^{n-k}}{\binom{n+k}{2k}} \frac{1}{n+k+1}$$

$$I_n^{(0)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\binom{2k}{k}} \frac{2^{n+k}}{2k+1}, \qquad I_n^{(0)} = \frac{1}{\binom{2n}{n}} \frac{1}{2n+1} \sum_{k=0}^{n} \binom{2k}{k} 2^{2n-k}.$$

Let $\varphi = (1+\sqrt{5})/2$ and $\widehat{\varphi} = (1-\sqrt{5})/2$. We have $\varphi^2 = \varphi + 1$ and $\widehat{\varphi}^2 = \widehat{\varphi} + 1$. Moreover, by the Binet formula, we have that the Fibonacci and Lucas numbers are defined by $F_n = (\varphi^n - \widehat{\varphi}^n)/\sqrt{5}$ and $L_n = \varphi^n + \widehat{\varphi}^n$. If $s = 2\varphi$, then $s+2 = 2(\varphi+1) = 2\varphi^2$ and $4 - s^2 = 4(1-\varphi^2) = -4\varphi$. Similarly for $s = 2\widehat{\varphi}$. Now, let $A_n = (I_n^{(2\varphi)} - I_n^{(2\widehat{\varphi})})/\sqrt{5}$ and $B_n = I_n^{(2\varphi)} + I_n^{(2\widehat{\varphi})}$. Using formulas (4), (7), (11) and (20), it is straightforward to obtain the identities

$$\begin{aligned}
A_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k}}{2k+1} (F_{n+3k+2} - F_{n+k+1}) \\
A_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{2^{n-k}}{\binom{n+k}{2k}} \frac{1}{n+k+1} F_{2n-2k} \\
A_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\binom{2k}{k}} \frac{2^k}{2k+1} (2^n F_{2n+2k+2} - 2^k F_{2k+1}) \\
A_n &= \frac{1}{\binom{2n}{n}} \frac{1}{2n+1} \sum_{k=0}^{n} \binom{2k}{k} (-4)^{n-k} (2^k F_{n+k+2} - F_{n-k+1})
\end{aligned}$$

and the analogues ones for the Lucas numbers.

6 Log-behavior of the sequence $I_n^{(s)}$

We now consider the rate of growth of the sequence $\{I_n^{(s)}\}_{n\geq 0}$ when the integrals are all positive. Recall that a sequence $\{a_n\}_{n\geq 0}$ of positive real numbers is *log-concave* (or *logarithmically concave*) when the sequence $\{a_{n+1}/a_n\}_{n\geq 0}$ is decreasing, or, equivalently, when $a_n^2 \geq a_{n-1}a_{n+1}$ for every $n \geq 1$, and is *log-convex* (or *logarithmically convex*) when the sequence

$\{a_{n+1}/a_n\}_{n\geq 0}$ is increasing, or, equivalently, when $a_n^2 \leq a_{n-1}a_{n+1}$ for every $n \geq 1$. Moreover, the sequence $\{a_n\}_{n\geq 0}$ is *concave* when $2a_n \geq a_{n-1}+a_{n+1}$ for every $n \geq 1$, and is *convex* when $2a_n \leq a_{n-1} + a_{n+1}$ for every $n \geq 1$. By the arithmetic-geometric mean inequality, the log-convexity implies the convexity. These properties find application in many fields, especially in combinatorics [4, 8, 18].

In our case, we have the following theorem.

Theorem 7. *For $s \geq -2$, the sequence $\{I_n^{(s)}\}_{n\geq 0}$ is log-convex (and convex). Moreover*

$$\lim_{n\to+\infty} \frac{I_{n+1}^{(s)}}{I_n^{(s)}} = s+2 \qquad \textit{and} \qquad \lim_{n\to+\infty} \sqrt[n]{I_n^{(s)}} = s+2\,. \tag{22}$$

Proof. By Lemma 2.3 in [4], given a positive continuous function $f : [a,b] \to \mathbb{R}$ and the integrals $I_n = \int_a^b f(x)^n \mathrm{d}x$ for $n \geq 0$, we have that the sequence $\{I_n\}_{n\geq 0}$ is log-convex. In our case, we have the continuous function $f : [0,1] \to \mathbb{R}$ given by $f(x) = 1 + sx + x^2$. Since this function is positive for $s > -2$, the sequence $\{I_n^{(s)}\}_{n\geq 0}$ is log-convex for $s > -2$. For $s = -2$, we have $I_n^{(-2)} = \frac{1}{2n+1}$ and this sequence is log-convex. Notice that, for $s \geq -2$, the integrals $I_n^{(s)}$ are positive for all $n \in \mathbb{N}$.

Moreover, as proved in [4, Formula 4.41], if a sequence $\{y_n\}_{n\geq 0}$ of non-zero real numbers satisfy a linear recurrence of the first order $y_{n+1} = a_{n+1}y_n + b_{n+1}$, with $a_n \neq 0$ for $n \geq 1$ and $b_n \neq 0$ for $n \geq 0$, then the sequence $\{q_n\}_{n\geq 0}$ of the quotients $q_n = y_{n+1}/y_n$ satisfy the recurrence

$$q_{n+1} = a_{n+1} + \frac{b_{n+1}}{b_n} - a_n \frac{b_{n+1}}{b_n}\frac{1}{q_n}\,. \tag{23}$$

Since the integrals $I_n^{(s)}$ satisfy recurrence (18), the quotients $q_n = I_{n+1}^{(s)}/I_n^{(s)}$ satisfy recurrence (23) with the coefficients given in (21).

Let $q = \lim_{n\to+\infty} q_n$ and observe that

$$\lim_{n\to+\infty} a_n = \frac{4-s^2}{4} \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{b_{n+1}}{b_n} = s+2\,.$$

So, taking the limit of equation (23) as $n \to +\infty$, we obtain

$$q = \frac{4-s^2}{4} + s + 2 - \frac{4-s^2}{4}\,(s+2)\,\frac{1}{q}$$

that is

$$4q^2 + (s+2)(s-6)q - (s-2)(s+2)^2 = 0\,.$$

This equation has two roots: $q_1 = s+2$ and $q_2 = -(s+2)(s-2)/8 < 0$. Since $q > 0$, we have the first limit in (22). Finally, this first limit implies the second one [7, p. 57]. □

Furthermore, we have

Theorem 8. *For $s > -2$, the series $\sum_{n\geq 0} I_n^{(s)}\, z^n$ converges for $|z| < \frac{1}{s+2}$. Moreover, for $s > -1$, this series converges also for $z_0 = -\frac{1}{s+2}$ and*

$$\sum_{n\geq 0} (-1)^n \frac{I_n^{(s)}}{(s+2)^n} = \frac{2(s+2)}{\sqrt{12+4s-s^2}} \arctan \frac{\sqrt{12+4s-s^2}}{3(s+2)}\,. \tag{24}$$

Proof. From limits (22), we have at once the disk of convergence of the given series. For the second part, consider the series $\sum_{n\geq 0}(-1)^n a_n$ where $a_n = \frac{I_n^{(s)}}{(s+2)^n}$. Since the coefficients a_n are positive, we have an alternating series. Moreover, the sequence $\{a_n\}_{n\geq 0}$ is decreasing, being $I_{n+1}^{(s)} \leq (s+2) I_n^{(s)}$. Finally $a_n \to 0$ as $n \to 0$. Indeed, from recurrence (18), we obtain the identity

$$a_{n+1} = \frac{2-s}{2}\,\frac{n+1}{2n+3}\, a_n + \frac{1}{2(2n+3)} \left(s + 2 - \frac{s}{(s+2)^{n+2}} \right).$$

Now, setting $A = \lim_{n\to+\infty} a_n$ and taking the limit of this equation as $n \to +\infty$, we obtain $(s+2)\,A = 0$ (since $s > -1$), that is $A = 0$. In conclusion, by the Leibniz criterion, the considered series converges and, by Abel's theorem, we have

$$\sum_{n\geq 0} (-1)^n \frac{I_n^{(s)}}{(s+2)^n} = \lim_{t\to z_0^-} I^{(s)}(t) = I^{(s)}\Big(-\frac{1}{s+2}\Big)\,.$$

Using (13), we have identity (24). □

For instance, for $s = 0$, identity (24) becomes

$$\sum_{n\geq 0} (-1)^n \frac{I_n^{(0)}}{2^n} = \frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}\,.$$

From limits (22), we also have the following property.

Theorem 9. *For $s \geq -2$, the series $\sum_{n\geq 0} \frac{z^n}{I_n^{(s)}}$ converges for $|z| < s+2$.*

For instance, we have

$$\sum_{n\geq 0} \frac{1}{I_n^{(1)}} \approx 2.02811711\,, \qquad \sum_{n\geq 0} \frac{(-1)^k}{I_n^{(1)}} \approx 0.63877409\,.$$

Recall that a sequence $\{a_n\}_{n\geq 0}$ of positive real numbers is *star shaped* when $\frac{a_n}{n} \leq \frac{a_{n+1}}{n+1}$ for all $n \geq 1$. By Lemma 1 in [19], if a sequence $\{a_n\}_{n\geq 0}$ is convex and $a_1 \leq a_2/2$, then it is star shaped. This immediately implies

Theorem 10. *For $s \geq \frac{1}{20}\left(\sqrt{1185}-15\right) \approx 0.971191$, the sequence $\{I_n^{(s)}\}_{n\geq 0}$ is star shaped.*

Notice that if a sequence $\{a_n\}_{n\geq 0}$ is log-convex and star shaped, then $a_n^2 \leq a_{n-1}a_{n+1}$ for $n \geq 1$ and $a_{n-1} \leq \frac{n-1}{n}\, a_n$ for $n \geq 2$. Hence $a_n^2 \leq \frac{n-1}{n}\, a_n a_{n+1}$, that is $a_n \leq \frac{n-1}{n}\, a_{n+1}$ for $n \geq 2$. Repeating this procedure starting from the inequalities $a_n^2 \leq a_{n-1}a_{n+1}$ for $n \geq 1$ and $a_n \leq \frac{n-1}{n}\, a_{n+1}$ for $n \geq 2$, we have $a_n \leq \frac{n-2}{n-1}\, a_{n+1}$ for $n \geq 3$. Continuing in this ways, we have the following

Lemma 11. *If a sequence $\{a_n\}_{n\geq 0}$ of positive real numbers is log-convex and star shaped, then*

$$a_n \leq \frac{n-m}{n-m+1}\, a_{n+1} \qquad \text{for every } m, n \in \mathbb{N},\ n \geq m+1\,.$$

In conclusion, by Theorems 7 and 10 and by Lemma 11, we have, for $s \geq \frac{1}{20}\left(\sqrt{1185}-15\right)$, the inequalities

$$\frac{I_n^{(s)}}{n-m} \leq \frac{I_{n+1}^{(s)}}{n-m+1} \qquad \text{for every } m, n \in \mathbb{N},\ n \geq m+1\,.$$

7 Final remarks: Appell polynomials

In this final section, we consider the polynomials defined by

$$I_n^{(s)}(x) = \int_0^1 (x + su + u^2)^n \,\mathrm{d}u \qquad (s \in \mathbb{R})\,. \tag{25}$$

Clearly, they are an extension of the integrals $I_n^{(s)} = I_n^{(s)}(1)$. Moreover, as can be easily verified, they form an Appell sequence [1, 10], that is a polynomial sequence $\{a_n(x)\}_{n\geq 0}$ such that $\deg a_n(x) = n$ and $a_n'(x) = na_{n-1}(x)$ for every $n \in \mathbb{N}$. Equivalently, an Appell sequence is a polynomial sequence $\{a_n(x)\}_{n\geq 0}$ with exponential generating series

$$\sum_{n\geq 0} a_n(x)\frac{t^n}{n!} = g(t)\,\mathrm{e}^{xt}$$

for a given exponential series $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ with $g_0 \neq 0$. The class of Appell sequences contains several classical polynomials (such as the Hermite polynomials, the Bernoulli and Euler polynomials and the rencontres polynomials). Moreover, the Appell sequences belong to the larger class of Sheffer sequences, which have several applications in combinatorics [9, 10, 11] and in the modern umbral calculus [14, 15, 16].

Expanding the power in (25) in the natural way, it is possible to find the following expression

$$I_n^{(s)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} \frac{s^{n-k-i}}{n-k+i+1} \right) x^k .$$

Moreover, several of the results obtained for the integrals $I_n^{(s)}$ can be extended straightforwardly to the polynomials $I_n^{(s)}(x)$. For instance, formula (4) can be extended to

$$I_n^{(s)}(x) = \frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(s+2)^{2k+1} - s^{2k+1}}{2k+1} (4x - s^2)^{n-k} .$$

This identity implies at once that the exponential generating series for our polynomials is

$$\sum_{n\geq 0} I_n^{(s)}(x)\frac{t^n}{n!} = g^{(s)}(t)\,\mathrm{e}^{xt}$$

where

$$g^{(s)}(t) = \mathrm{e}^{-\frac{s^2}{4}t} \sum_{k\geq 0} \frac{(s+2)^{2k+1} - s^{2k+1}}{2^{2k+1}(2k+1)} \frac{t^k}{k!} .$$

On the other hand, the ordinary generating series turns out to be

$$\sum_{n\geq 0} I_n^{(s)}(x)\, t^n$$

$$= \frac{2}{\sqrt{-t}\sqrt{4+(s^2-4x)t}}\left(\arctan\frac{(s+2)\sqrt{-t}}{\sqrt{4+(s^2-4x)t}} - \arctan\frac{s\sqrt{-t}}{\sqrt{4+(s^2-4x)t}}\right)$$

and formula (11) can be extended to

$$I_n^{(s)}(x) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{\binom{2k}{k}}\frac{1}{2k+1}\left((s+2)^{2k+1}(x+s+1)^{n-k} - s^{2k+1}x^{n-k}\right).$$

Similarly, recurrence (18) can be extended to the recurrence

$$I_{n+1}^{(s)}(x) = \frac{4x-s^2}{2}\,\frac{n+1}{2n+3}\,I_n^{(s)}(x) + \frac{(s+2)(x+s+1)^{n+1} - sx^{n+1}}{2(2n+3)}$$

and consequently formula (20) can be extended to

$$I_n^{(s)}(x) = \frac{1}{2}\frac{1}{\binom{2n}{n}}\frac{1}{2n+1}\sum_{k=0}^{n}\binom{2k}{k}\left((s+2)(x+s+1)^k - sx^k\right)(4x-s^2)^{n-k}.$$

Finally, notice that the polynomials $I_n^{(s)}(x)$ have all the properties of the Appell polynomials. For instance, we have

$$I_n^{(s)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} I_k^{(s)}(x)\, y^{n-k}.$$

In particular, we have

$$I_n^{(s)}(x+1) = \sum_{k=0}^{n}\binom{n}{k} I_k^{(s)}(x) \qquad \text{or} \qquad I_n^{(s)}(x+1) = \sum_{k=0}^{n}\binom{n}{k} I_{n-k}^{(s)}\, x^k.$$

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