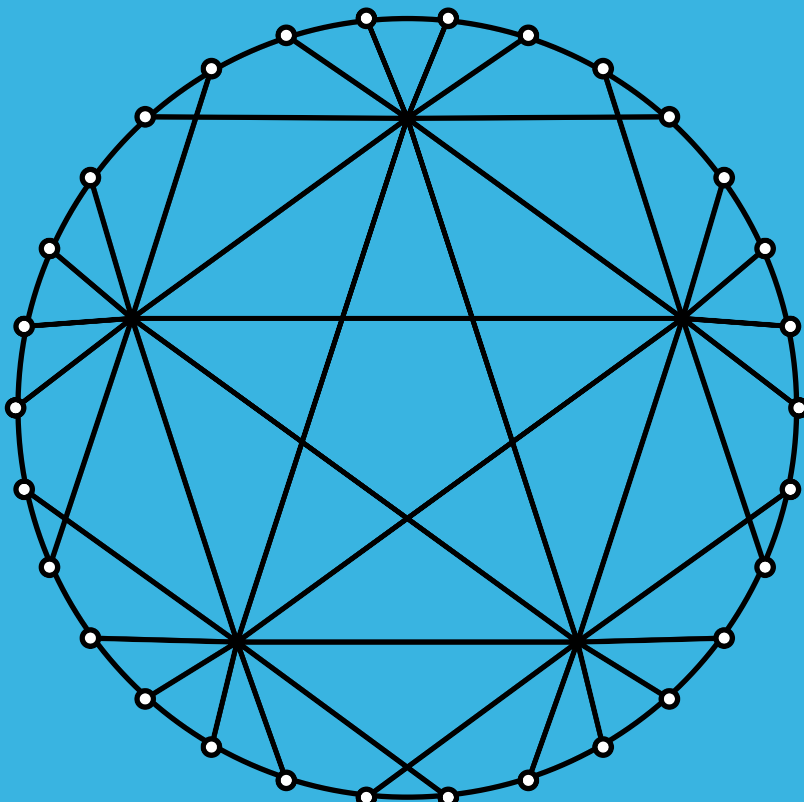
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 96
October 2022



Duluth, Minnesota, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **96** (2022), 68–83

Distance quasi-magic regular graphs and some applications

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Abstract. We introduce the following variation of distance magic labeling. Let G be a graph of order n and ℓ a bijection from the vertex set of G to a set of labels $\{1, 2, \ldots, \varsigma-1, \varsigma+1, \ldots, n+1\}$ for some $\varsigma \in \{2, \ldots, n\}$. The weight of a vertex is defined as the sum of the labels of all neighboring vertices. If the weight of every vertex is equal to the same fixed constant, then we say ℓ is a distance quasi-magic labeling of G. This differs from a distance magic labeling in which the first n positive integers are used as labels. Our main results include constructing r-regular distance quasi-magic graphs of order n for some pairs (n, r) such that no r-regular distance magic graph of order n exists. To do this, we use a mix of established tools as well as our own. Then we apply these results in three different areas: construction of quasi-equalized incomplete tournaments, face-magic labelings of type (a, b, c) for Dutch windmill graphs, and cycle-magic labelings of generalized book graphs.

1 Introduction

Let G be a simple graph and $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For a vertex $v \in V$, define the *weight* of v as

$$w(v) = \sum_{uv \in E(G)} f(u).$$

If there exists some constant μ, called the *magic constant*, such that $w(v) = \mu$ for all $v \in V(G)$, we say that f is a *distance magic labeling* of G. Further,

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Key words and phrases: distance magic labeling, distance quasi-magic graphs, nearly distance magic, face-magic labeling of type (a, b, c), H-magic labelings, equalized incomplete tournaments
AMS (MOS) Subject Classifications: 05C78

Received: 30 December 2021
Accepted: 3 April 2022

a graph that admits a distance magic labeling is called a *distance magic graph.* For surveys of results in this area, we direct the reader to [1] or [5].

The focus of this manuscript is on regular graphs which are not distance magic graphs. We review some pertinent results next.

Theorem 1.1 (Vilfred [12])**.** *Let G be an r-regular graph. If r is odd, then G is not a distance magic graph.*

Let $K_{p:n} \cong K_{n,n,\ldots,n}$ (n appears p times in the subscript) denote the complete equipartite graph where each of the p partite sets contain n vertices. It is easy to observe that $K_{p:n}$ is distance magic if and only if there exists a partition of the set $S = \{1, 2, \ldots, pn\}$ into p sets of size n such that the sum of the elements in each of the sets is constant. Such a partition is known as a *constant sum partition.* Miller, Rodger, and Simanjuntak proved the following in [11].

Theorem 1.2 (Miller et al. [11])**.** *Let $n, p > 1$. The complete equipartite graph $K_{p:n}$ is distance magic if and only if n is even or both n and p are odd.*

Regarding even-regular graphs, Froncek, Kovar, and Kovarova proved the following in [3].

Theorem 1.3 (Froncek et al. [3])**.** *Let $n \geq 4$ be an even integer. There exists a distance magic r-regular graph G of order n if and only if r is even, $2 \leq r \leq n-2$, and $n \equiv 0 \pmod 4$ or $r \equiv 0 \pmod 4$.*

Theorems 1.1 and 1.3 lead to the following non-existence corollary for graphs of singly even order.

Corollary 1.4. *Let G be an r-regular graph of order n. If $n \equiv 2 \pmod 4$ and $r \not\equiv 0 \pmod 4$, then G is not a distance magic graph.*

Corollary 1.4 provides the motivation for this manuscript. We wonder: Can one achieve something akin to a distance magic labeling of an r-regular graph of order n when n is singly even and r is not divisible by 4?

This question was addressed by Godinho, Singh, and Arumugam in [6]. They relaxed the requirement that every vertex has the same weight, and instead asked that the weight of every vertex is either k or $k+1$ for some k. They called the labeling a *nearly distance magic labeling* and proved some basic results.

We take an approach more in line with Froncek, Paramasivam, and Prajeesh who recently introduced the notion of a quasimagic rectangle in [4] (this will be discussed more in Section 2). We propose the following labeling which asks that the weight of each vertex is constant (as with distance magic) but allows one label to be excluded.

Definition 1.5. Let G be a simple graph of order n and $\ell : V(G) \to \{1, 2, \ldots, \varsigma-1, \varsigma+1, \ldots, n+1\}$ be a bijection for some integer $\varsigma \in \{2, \ldots, n\}$. If there exists a fixed constant μ called the *magic constant* such that

$$w(v) = \sum_{uv \in E(G)} \ell(u) = \mu$$

for every $v \in V(G)$, then ℓ is a *distance quasi-magic labeling* and the graph G is a *distance quasi-magic graph.*

Figure 1 shows two different distance quasi-magic labelings of $K_{3,3}$, one with $\varsigma = 4$ and $\mu = 12$, and one with $\varsigma = 6$ and $\mu = 11$. Note that $K_{3,3}$ is not a distance magic graph.

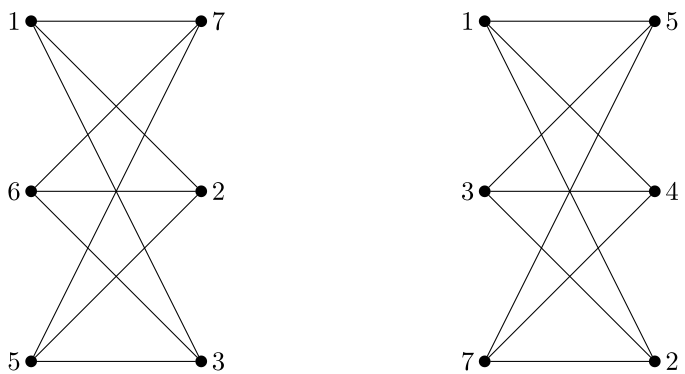

Figure 1: Two different distance quasi-magic labelings of $K_{3,3}$

The following can be said about the necessary conditions for r-regular distance quasi-magic graphs.

Lemma 1.6. *Let G be an r-regular graph of order n with distance quasi-magic labeling $\ell : V(G) \to \{1, 2, \ldots, \varsigma-1, \varsigma+1, \ldots, n+1\}$. Then the magic constant of ℓ is*

$$\mu = \frac{r((n+1)(n+2)/2 - \varsigma)}{n}.$$

Proof. Let $L = \{1, 2, \ldots, \varsigma - 1, \varsigma + 1, \ldots, n + 1\}$ and

$$w(G) = \sum_{v \in V(G)} w(v).$$

On one hand, $w(G) = n\mu$. On the other hand,

$$\begin{aligned} w(G) &= r \textstyle\sum_{i \in L} i \\ &= r(\tfrac{(n+1)(n+2)}{2} - \varsigma). \end{aligned}$$

Hence, $\mu = \frac{r((n+1)(n+2)/2-\varsigma)}{n}$, as claimed. □

In the sections that follow we construct distance quasi-magic labelings of r-regular graphs of order n for pairs (n, r) such that there exists no r-regular distance magic graph of order n. We then provide three disparate applications for distance quasi-magic labeling; one to tournament design, and two involving constructions of face-magic or cycle-magic labelings.

2 Tools

An $a \times b$ *magic rectangle* $MR(a, b)$ is an a by b array of the integers $1, 2, \ldots, ab$ such that no integer is repeated, the sum of all the integers in each row is equal to some constant ρ, and the sum of all the integers in each column is equal to some constant σ. It is easy to see that $\rho = \frac{b(ab+1)}{2}$ and $\sigma = \frac{a(ab+1)}{2}$. Harmuth proved that such an array exists whenever $a \equiv b \pmod 2$ with the trivial exception when exactly one of a or b is 1 or $a = b = 2$ [7, 8]. Very recently, the nonexistence of even by odd (or odd by even) magic rectangles motivated Froncek, Paramasivam, and Prajeesh to introduce the following close relative of magic rectangles [4].

Let a be an odd integer and b an even integer. An $a \times b$ *quasi-magic rectangle* $QMR(a, b : \varsigma)$ is an a by b array such that each of the integers

$1, 2, \ldots, \varsigma - 1, \varsigma + 1, \ldots, ab + 1$ appears exactly once, the sum of the entries in each row is equal to some constant ρ, and the sum of the entries in each column is equal to some constant σ.

Theorem 2.1 (Froncek et al. [4]). *A quasi-magic rectangle $QMR(a, 2t : at + 1)$ exists for all odd $a \geq 1$ and $t \geq 1$ except when $t = 1$ and $a \equiv 1 \pmod 4$.*

Though a $QMR(a, 2 : \varsigma)$ does not exist if $a \equiv 1 \pmod 4$, the next lemma shows that it is possible to partition $[1, 2a + 1] \setminus \{a + 1\}$ into two sets of size a that have the same sum.

Lemma 2.2. *Let $a > 1$ and $a \equiv 1 \pmod 4$. Then the set $[1, 2a+1] \setminus \{a+1\}$ can be partitioned into two sets A and B such that $|A| = |B| = a$ and $\sum_{i \in A} i = \sum_{j \in B} j = a(a + 1)$*

Proof. We assign the members of A and B in a serpentine fashion with the exception of the column marked by $*$.

$$\begin{array}{ccccccccccc}
A & 1 & 4 & 5 & \ldots & a & a+4 & a+6 & a+7 & a+10 & \ldots & 2a+1 \\
B & 2 & 3 & 6 & \ldots & a+2 & a+3 & a+5 & a+8 & a+9 & \ldots & 2a \\
 & & & & & & & * & & & &
\end{array}$$

By comparing the columns, notice

$$\begin{array}{rcl}
\sum_{i \in A} i - \sum_{j \in B} j & = & (-1 + 1 - 1 + \cdots + 1) \\
 & & +(-2 + 1 + 1) \\
 & & +(-1 + 1 - 1 + \cdots + 1) \\
 & = & 0,
\end{array}$$

since there are an odd number of columns. Hence, $\sum_{i \in A} i = \sum_{j \in B} j$. The fact that this sum equals $a(a + 1)$ is an easy counting exercise and we leave it to the reader. $\square$

To construct 6-regular distance quasi-magic graphs in the next section, we will need a partition of the integers into sets of size 2 or 3 such that the sum of the elements in any two subsets of the same size is the same.

Lemma 2.3. *Let $n \equiv 2 \pmod 4$, $n \geq 26$, and $S = \{1, 2, \ldots, \frac{n}{2}, \frac{n}{2} + 2, \ldots, n + 1\}$. There exists a partition of S such that $S = A_1 \cup \cdots \cup A_{\frac{n}{2}-9} \cup B_1 \cup \cdots \cup B_6$, $|A_i| = 2$, $|B_j| = 3$, $\sum_{a \in A_i} a = n + 2$, and $\sum_{b \in B_j} b = 3(\frac{n}{2} + 1)$ for all i and j.*

Proof. The partition is

$$A_i = \{i, n+2-i\}$$

for $i = 1, 2, \ldots, \frac{n}{2} - 9$, and

$$\begin{aligned}
B_1 &= \{\tfrac{n}{2} - 8, \tfrac{n}{2} + 2, \tfrac{n}{2} + 9\},\\
B_2 &= \{\tfrac{n}{2} - 7, \tfrac{n}{2} + 3, \tfrac{n}{2} + 7\},\\
B_3 &= \{\tfrac{n}{2} - 6, \tfrac{n}{2} + 4, \tfrac{n}{2} + 5\},\\
B_4 &= \{\tfrac{n}{2} - 5, \tfrac{n}{2} - 2, \tfrac{n}{2} + 10\},\\
B_5 &= \{\tfrac{n}{2} - 4, \tfrac{n}{2} - 1, \tfrac{n}{2} + 8\},\\
B_6 &= \{\tfrac{n}{2} - 3, \tfrac{n}{2}, \tfrac{n}{2} + 6\}.
\end{aligned}$$

□

The *lexicographic product* of a graph G with a graph H, denoted $G \circ H$ or $G[H]$, can be constructed by replacing each vertex of G with a copy of H and replacing every edge of G with a complete bipartite graph between the vertices of the two corresponding copies of H. For example, $K_{m,m} \cong K_2 \circ \overline{K_m}$. We use the lexicographic product in the proofs contained in the next section.

3 Main results

In this section, we construct r-regular distance quasi-magic graphs of order n. Along the way we emphasize pairs (n, r) such that no r-regular distance magic graph of order n exists. We will call such a pair a *quasi pair*. We focus on quasi pairs (n, r) such that $n \equiv 2 \pmod 4$ and $r \not\equiv 0 \pmod 4$. In particular, we show that $(n, 6)$ is a quasi pair whenever $n \equiv 2 \pmod 4$ and $n = 18$ or $n \geq 26$. In addition, we also show that if $n = ab$ where $a > 1$ is odd and $b \equiv 2 \pmod 4$, then $(n, (2i-1)a)$ and $(n, (4j-2)a)$ for $i \in [1, b-1]$ and $j \in [1, \frac{b-2}{4}]$ are quasi pairs. We begin by describing a construction using quasi-magic rectangles.

Theorem 3.1. *Let a be odd, b even, and $a, b > 1$. There exists an r-regular distance quasi-magic graph of order $n = ab$ for every $r \in \{a, 2a, \ldots, (b-1)a\}$.*

Proof. Let $r' \in \{1, 2, \ldots, b-1\}$ and G' be any r'-regular graph of order b. Form the graph $G \cong G' \circ \overline{K_a}$ with vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_b$ where each V_i is a set of a isolated vertices corresponding to the appearance of $\overline{K_a}$ in the product. G is $(r'a)$-regular and has order $n = ab$.

If $a \equiv 1 \pmod 4$ and $b = 2$, let f and g, respectively be arbitrary bijections between V_1 and A, V_2 and B, respectively for A and B given by Lemma 2.2. Otherwise, assign the labels of $V(G)$ using a quasi-magic rectangle as follows. For $1 \leq i \leq b$, let f_i be an arbitrary bijection between V_i and column i of a quasi-magic rectangle $QMR(a, b : \varsigma)$. It is easy to see that each vertex of G has been assigned exactly one of the labels in $\{1, 2, \ldots, \varsigma - 1, \varsigma + 1, \ldots, n + 1\}$ and for every $v \in V(G)$,

$$w(v) = r'\sigma$$

where $\sigma = a(a + 1)$ or σ is the magic column sum of the $QMR(a, b : \varsigma)$. Hence, we have described a distance quasi-magic labeling of G. $\square$

The fact that $(18, 6)$ is a quasi pair follows easily from the last theorem. In addition, we have the following corollary for complete equipartite graphs.

Corollary 3.2. *Let $n, p > 1$. The complete equipartite graph $K_{p:n}$ is a distance quasi-magic graph if n is odd and p is even.*

Together, Theorem 1.2 and Corollary 3.2 say that every complete equipartite graph is distance magic or distance quasi-magic. Also, many of the pairs given by Theorem 3.1 are quasi pairs. For example, $(6, 3)$ is a quasi pair and a distance quasi-magic graph with these parameters is shown in Figure 1.

The next result shows that $(n, 6)$ is a quasi pair for any $n \equiv 2 \pmod 4$ and $n \geq 26$.

Theorem 3.3. *Let $n \equiv 2 \pmod 4$ and $n \geq 26$. There exists a 6-regular distance quasi-magic graph of order n.*

Proof. Let G be any 3-regular graph of order $\frac{n}{2} - 9$ and H be any 2-regular graph of order 6. We claim the graph $\mathcal{G} \cong (G \circ \overline{K_2}) \cup (H \circ \overline{K_3})$ is a distance quasi-magic graph. Indeed, Lemma 2.3 provides the correct labeling in the following way. Arbitrarily assign each pair of blown up vertices in $G \circ \overline{K_2}$ the labels from the set A_i for $i = 1, 2, \ldots, \frac{n}{2} - 9$, and each triple of blown up vertices in $H \circ \overline{K_3}$ the labels from the set B_j for $j = 1, 2, \ldots, 6$. Since this assignment is a bijection from $V(\mathcal{G})$ to $\{1, 2, \ldots, \frac{n}{2}, \frac{n}{2} + 2, \ldots, n + 1\}$, it only remains to check the weight of each vertex. Let $v \in V(\mathcal{G})$. If $v \in V(G \circ \overline{K_2})$, then

$$w(v) = 3(n+2).$$

On the other hand, if $v \in V(H \circ \overline{K_3})$, then

$$\begin{aligned} w(v) &= 2 \cdot 3(\tfrac{n}{2}+1) \\ &= 3(n+2), \end{aligned}$$

so we have proved the claim. □

4 Applications

In this section we describe three applications for distance quasi-magic labelings. The first is a variation of equalized incomplete tournaments which equalize the total opponent strength for each competitor. The second application is to face-magic labelings of Dutch windmill graphs, and the third is to cycle-magic labelings of generalized book graphs.

4.1 Tournaments

An equalized incomplete tournament $EIT(n,r)$ consists of n competitors ranked by strength 1 through n, where each competitor plays r matches against r distinct opponents and the sum of the rankings of all r opponents to be played is some fixed constant k for every competitor. It is known that distance magic graphs directly correspond to equalized incomplete tournaments [3].

Observation 4.1. *An r-regular distance magic graph of order n exists if and only if an equalized incomplete tournament $EIT(n,r)$ exists.*

Corollary 4.2. *If $n \equiv 2 \pmod 4$, there does not exist an equalized incomplete tournament $EIT(n,r)$ if $r \equiv 1, 2, or\, 3 \pmod 4$.*

Corollary 4.2 is analogous to Corollary 1.4 in that it provides the motivation for achieving something akin to equalized incomplete tournaments. We address this by introducing the following type of tournament motivated by distance quasi-magic labelings.

Suppose there is a tournament involving n competitors ranked by strength from 1 through $n+1$ excluding some integer $\varsigma \in [2, n]$, and each competitor plays exactly r matches against r distinct opponents. If for each competitor, the sum of the rankings of all r opponents to be played is some fixed constant k, we call this a *quasi-equalized incomplete tournament* $QEIT(n, r, \varsigma)$.

Observation 4.3. *An r-regular distance quasi-magic graph of order n exists if and only if a quasi-equalized incomplete tournament $QEIT(n, r, \varsigma)$ exists for some ς.*

We obtain the following corollaries from Theorems 3.1 and 3.3, respectively.

Corollary 4.4. *Let $a, b > 1$. If a is odd and b is even, then there exists a quasi-equalized incomplete tournament $QEIT(ab, r, \varsigma)$ for $r = a, 2a, \ldots, (b-1)a$ and some ς.*

Corollary 4.5. *Let $n \geq 26$ and $n \equiv 2 \pmod 4$. Then there exists a quasi-equalized incomplete tournament $QEIT(n, 6, \frac{n}{2})$.*

4.2 Face-magic labelings of Dutch windmills

Let $G = (V, E, F)$ be a planar graph and $a, b, c \in \{0, 1\}$. A labeling of type (a, b, c) is an assignment of a, b, and c labels to the vertices, edges, and faces of a graph, respectively, that uses each of the integers in $\{1, 2, \ldots, a|V| + b|E| + c|F|\}$ exactly once. The weight w of a face is calculated as the sum of the label of the face (if present) with the labels of the surrounding vertices and edges (when present) of the face. If there exists a fixed constant $\mu(s)$ such that the weight of every s-sided face is $\mu(s)$, then we say the labeling is a *face-magic labeling of type* (a, b, c) and call G a *face-magic graph of type* (a, b, c).

Classifying all triples (a, b, c) such that a graph admits a face-magic labeling of type (a, b, c) is known as the spectrum problem. Freyberg solved the spectrum problem for fans and subdivided fans, wheels and subdivided wheels, ladders and subdivided ladders, and chained cycles in [2]. Using distance magic and distance quasi-magic labelings in conjunction, we will solve the spectrum problem for the Dutch windmill graph.

The Dutch windmill graph D_n^m consists of m copies of the cycle graph C_n joined by a common vertex we refer to as the hub h. The graph D_3^m is also known as the friendship graph. Figure 2 shows the Dutch windmill graph

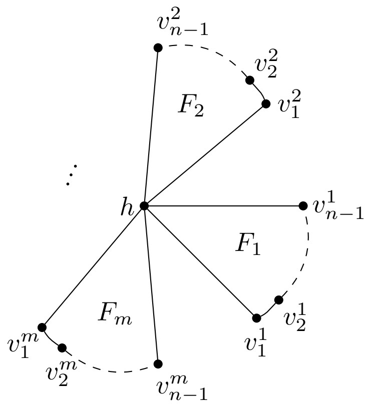

Figure 2: The Dutch windmill graph D_n^m

D_n^m equipped, for reference, with the labels used in the proof of Theorem 4.7.

In 1983, Lih found face-magic labelings of types $(1,0,0)$ and $(1,1,0)$ for the friendship graphs [10].

Theorem 4.6 (Lih [10])**.** *The friendship graph D_3^m admits types $(1,0,0)$ and $(1,1,0)$ face-magic labelings for all $m \geq 2$.*

Our next result not only solves the spectrum problem for friendship graphs, complementing Theorem 4.6, but more generally solves the spectrum problem for Dutch windmills.

Theorem 4.7. *Let $n \geq 3$, $m \geq 2$, and $a,b,c \in \{0,1\}$. The Dutch windmill graph D_n^m is face-magic of type (a,b,c) for any triple (a,b,c) except when $(a,b,c) = (0,0,1)$ or m is even, n is odd, and $(a,b,c) = (0,1,0)$.*

Proof. Let $G \cong D_n^m = (V,E,F)$ be embedded in the plane according to its namesake. It is trivial that every graph is a type $(0,0,0)$ face-magic graph and every graph with at least two faces is not a type $(0,0,1)$ face-magic graph. Also observe that a type $(0,1,0)$ face-magic labeling of G exists if and only if the complete equipartite graph $K_{m:n}$ admits a distance magic labeling. Hence, Theorem 1.2 provides the necessary and sufficient conditions for this case: Either n must be even or both m and n must be odd. We will provide a labeling for each of the 5 remaining cases next.

Let $V = \bigcup_{j=1}^{m} V_j \cup \{h\}$ where $V_j = \{v_i^j : 1 \leq i \leq n-1\}$ and v_i^j is the i^{th} vertex in cycle j, $E = \bigcup_{j=1}^{m} E_j$ where $E_j = \{v_i^j v_{i+1}^j : 1 \leq i \leq n-2\} \cup \{hv_1^j, hv_{n-1}^j\}$, and $F = \bigcup_{j=1}^{m} F_j \cup F_\infty$ where each F_j is the j^{th} copy of C_n, and F_∞ is the exterior face (see Figure 2). Each F_j has n sides while F_∞ has mn sides. Since F_∞ is the only mn-sided face, we need only equalize the weights of the n-sided faces. Clearly, $|V| = m(n-1)+1, |E| = mn$, and $|F| = m+1$.

Let $H \cong K_{m:n'}$ for some $n' \geq 2$, and denote the j^{th} partite set of vertices in $V(H)$ as V_j^H so that $|V_j^H| = n'$ and $V(H) = \bigcup_{j=1}^{m} V_j^H$. Let f' be a distance magic (DM) labeling or distance quasi-magic (DQM) labeling of H with magic constant k. The existence of such a labeling is given by Theorem 1.2 or Corollary 3.2, respectively. Let $f : f'(V_j^H) \to S$ be an arbitrary bijection for the value of n' and set S given in the table below. Then define $f(F_\infty)$ or $f(h)$, when necessary (and depending on whether f' is DM or DQM), also as indicated in the table.

type	n'	S	$f(F_\infty)$	$f(h)$
$(0,1,1)$	$n+1$	$E_j \cup F_j$	$m(n+1)+1,\ f'$ DM $\varsigma,\ f'$ DQM	
$(1,0,0)$	$n-1$	V_j		$m(n-1)+1,\ f'$ DM $\varsigma,\ f'$ DQM
$(1,0,1)$	n	$V_j \cup F_j$	$mn+2$	$mn+1,\ f'$ DM $\varsigma,\ f'$ DQM
$(1,1,0)$	$2n-1$	$V_j \cup E_j$		$m(2n-1)+1,\ f'$ DM $\varsigma,\ f'$ DQM

For each case above, the weight of the n-sided face F_j is

$$w(F_j) = \begin{cases} k + f(F_\infty) & \text{type } (0,1,1) \\ k + f(h) & \text{types } (1,0,0) \text{ and } (1,1,0) \\ k + f(F_\infty) + f(h) & \text{type } (1,0,1) \end{cases}$$

for $j = 1, 2, \ldots, m$. Since f is a bijection and the weight of each face is independent of j, the labeling f is a type (a,b,c) face-magic labeling in each case. The fact that G admits a type $(1,1,1)$ face-magic labeling follows immediately from the labelings of types $(1,0,0)$ and $(0,1,1)$. Hence, we have proved the claim. □

4.3 Cycle–magic labeling of generalized book graphs

A labeling of type $(1,1,0)$ of a graph G is called *H-magic* if for every subgraph $H' \cong H$, the sum of the labels of vertices and edges contained in H' is equal to some fixed constant. As with the labelings discussed earlier, we refer to this sum as the weight of H'. If a graph G admits such a labeling, we say G is an *H-magic graph.* This notion differs from that of type $(1,1,0)$ face-magic labeling since it is not dependent on the embedding of the graph. We define the following generalization of H-magic labeling.

Definition 4.8. Let $G = (V, E)$ be a graph, $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a set of non-isomorphic subgraphs of G, and f a bijection from $V \cup E \to [1, |V| + |E|]$. If there exists some constant $k(i)$ such that

$$\sum_{x \in V(H) \cup E(H)} f(x) = k(i)$$

for every subgraph $H \cong H_i$ and every $i = 1, 2, \ldots, n$, then f is an *$\mathcal{H}$-magic labeling*, and G is a *$\mathcal{H}$-magic graph.*

The book graph B_m is commonly defined as $B_m \cong K_{1,m} \square K_2$, the Cartesian product of the star graph with a single edge. Every subgraph of B_m isomorphic to C_4 is called a page, so B_m contains m pages. Lladó and Moragas proved the following in [9].

Theorem 4.9 (Lladó and Moragas [9])**.** *The book graph $B_m \cong K_{1,m} \square K_2$ is C_4-magic if m is odd.*

It is worth noting that a book also contains 6-cycles (start at the top corner of the spine, trace your finger around the outside 3 edges of one page to the bottom of the spine, then trace your way back to the top corner of the spine along a different page). We will show that B_m is $\{C_4, C_6\}$-magic for all $m \geq 1$ and provide a similar result for a more generalized family which contain book graphs next.

The *generalized book graph* $GB(m, n)$ consists of m copies of the cycle graph C_n which all share a common edge called the spine. Therefore, $GB(m, 4) \cong B_m$. Figure 3 below shows the namesake embedding of the (generalized) book graph $GB(3, 4)$, while Figure 4 shows a planar embedding of a generalized book graph $GB(4, 5)$ along with a $\{C_5, C_8\}$-magic labeling. In total, $GB(m, n)$ has $m(n-2)+2$ vertices, $m(n-1)+1$ edges, and it is easy to see that if H is a cycle and a subgraph of $GB(m, n)$, then $H \cong C_n$ or $H \cong C_{2n-2}$.

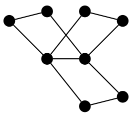

Figure 3: The (generalized) book graph $GB(3,4) \cong B_3$

Theorem 4.10. *The generalized book graph $GB(m,n)$ is $\{C_n, C_{2n-2}\}$-magic for every $m \geq 1$ and $n \geq 3$.*

Proof. Let $G \cong GB(m,n) = (V,E)$ with $V = \{v_i^j : 1 \leq i \leq n-2, 1 \leq j \leq m\} \cup \{s_1, s_2\}$ where v_i^j is the i^{th} vertex in cycle j, s_1 and s_2 are the vertices on the spine, and $E = \{v_i^j v_{i+1}^j : 1 \leq i \leq n-3, 1 \leq j \leq m\} \cup \{s_1 v_1^j, s_2 v_{n-2}^j, 1 \leq j \leq m\} \cup \{s_1 s_2\}$. Let $H \cong K_{m:2n-3}$ and $V_i(H)$ be the i^{th} partite set of $V(H)$ for $i = 1, 2, \ldots, m$. We describe a bijection $f : V \cup E \to [1, m(2n-3)+3]$ as follows. There are two cases.

Case 1. m is odd.
Let f' be a distance magic labeling of H with magic constant k. Let f be any arbitrary bijection between $f'(V_j(H))$ and the vertices and edges of G given by $\{v_i^j : 1 \leq i \leq n-2\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq n-3\} \cup \{s_1 v_1^j, s_2 v_{n-2}^j\}$. Then let $f(s_1) = m(2n-3)+1$, $f(s_2) = m(2n-3)+2$, and $f(s_1 s_2) = m(2n-3)+3$.

Case 2. m is even.
Let f' be a distance quasi-magic labeling of H with magic constant k and excluded label ς. Let f be any arbitrary bijection between $f'(V_j(H))$ and the vertices and edges of G given by $\{v_i^j : 1 \leq i \leq n-2\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq n-3\} \cup \{s_1 v_1^j, s_2 v_{n-2}^j\}$. Then let $f(s_1) = \varsigma$, $f(s_2) = m(2n-3)+2$, and $f(s_1 s_2) = m(2n-3)+3$.

Let C be a subgraph of G isomorphic to a cycle. If $C \cong C_n$, then for some $j \in [1,m]$, the weight of C is

$$\begin{aligned} w(C) &= \textstyle\sum_{i=1}^{n-2} f(v_i^j) + \sum_{i=1}^{n-3} f(v_i^j v_{i+1}^j) + f(s_1 v_1^j) \\ &\quad + f(s_2 v_{n-2}^j) + f(s_1) + f(s_1 s_2) + f(s_2) \\ &= k + 2m(2n-3) + 5 + f(s_1) \\ &= \begin{cases} k + 3m(2n-3) + 6 & \text{if } m \text{ is odd} \\ k + 2m(2n-3) + 5 + \varsigma & \text{if } m \text{ is even} \end{cases}. \end{aligned}$$

Otherwise, if $C \cong C_{2n-2}$, then for some $j, j' \in [1,m]$ with $j \neq j'$, the weight

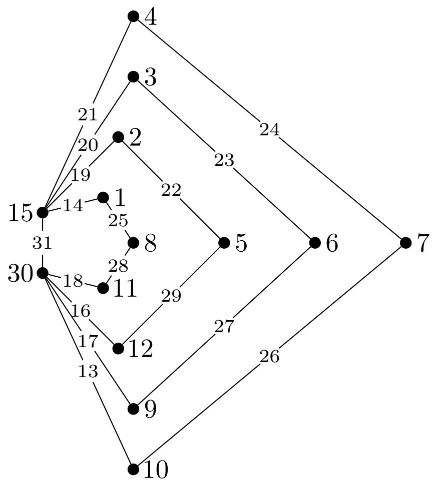

Figure 4: A $\{C_5, C_8\}$-magic labeling of $GB(4,5)$

of C is

$$\begin{aligned} w(C) &= \textstyle\sum_{i=1}^{n-2}(f(v_i^j)+f(v_i^{j'}))+\sum_{i=1}^{n-3}(f(v_i^j v_{i+1}^j)+f(v_i^{j'} v_{i+1}^{j'})) \\ &\quad +f(s_1)+f(s_1v_1^j)+f(s_1v_1^{j'})+f(s_2)+f(s_2v_{n-2}^j)+f(s_2v_{n-2}^{j'}) \\ &= 2k+f(s_1)+f(s_2) \\ &= \begin{cases} 2k+2m(2n-3)+3 & \text{if } m \text{ is odd} \\ 2k+m(2n-3)+2+\varsigma & \text{if } m \text{ is even} \end{cases}. \end{aligned}$$

Since f is a bijection, the weight of every subgraph of G isomorphic to C_n is equal to some fixed constant, and the weight of every subgraph isomorphic to C_{2n-2} is equal to some fixed constant, we have proved the theorem. $\square$

5 Concluding remarks

We defined a quasi pair as a pair (n,r) such that an r-regular distance quasi-magic graph of order n exists but an r-regular distance magic graph of order n does not exist. We constructed quasi pairs for $n \equiv 2 \pmod 4$ of the forms:

- $(n,6)$ whenever $n=18$ or $n \geq 26$.
- $(n,(2i-1)a)$ and $(n,(4j-2)a)$ whenever $n=ab$, $a>1$ is odd, and $b \equiv 2 \pmod 4$ for $i \in [1,b-1]$ and $j \in [1, \frac{b-2}{4}]$.

One direction forward is to complement the first item above by finding, or ruling out, the existence of distance quasi-magic labelings for 6-regular graphs of orders $n \in \{10, 14, 22\}$. Alternatively, since no odd-regular graph is distance magic, it might be interesting to explore distance quasi-magic labelings of odd-regular graphs with parameters other than those in the second item above.

Acknowledgements

This work was partially supported by the University of Minnesota Office of Undergraduate Research. The authors thank the anonymous referee for their careful reading of our manuscript and many insightful suggestions.

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